Qualifying Examination, Real Analysis
January 2006

Work each problem.

(1) Show that if $f \in L^p(\mathbb{R})$ and $1 \le p < \infty$ then $\int_n^{n+1} f(x)\ dx \to 0$ as $n \to \infty$.

(2) Say that the function f on $[0,1]$ is of bounded $\frac{1}{2}$ variation, and write $f \in BV_{1/2}$, if there is some positive constant C such that whenever $\{x_j\}_{j=0}^n$ is a partition of $[0,1]$, we have $\sum_{j=1}^n |f(x_j) - f(x_{j-1})|^{1/2} \le C$.
   (1) Show that $f \in BV_{1/2}$ implies that f is of bounded variation on $[0,1]$.
   (2) Show that any step function is in $BV_{1/2}$.
   (3) Show that $f \in BV_{1/2}$ implies $|f| \in BV_{1/2}$.
   (4) Show that there are no nonconstant continuous functions in $BV_{1/2}$. (Hint: If f is continuous and $|f(y_1)| < |f(y_2)|$, there is a partition $\{t_j\}_{j=0}^n$ of the interval with endpoints y_1 and y_2 satisfying $||f(t_j)| - |f(t_{j-1}|| = ||f(y_1)| - |f(y_2)||/n$ for all j.)

(3) The Weierstrass M-test states that

$$\sum_{k=1}^{\infty} \lim_{n\to\infty} x_{n,k} = \lim_{n\to\infty} \sum_{k=1}^{\infty} x_{n,k}$$

provided that $\lim_{n\to\infty} x_{n,k}$ exists for each k and $|x_{n,k}| \le M_k$ where $\sum_{k=1}^{\infty} M_k < \infty$. Obtain this result by using counting measure and dominated convergence.

(4) Let Ω be a countable set. Show that every σ-field $\mathcal{F}$ on Ω is generated by a partition. (Hint: For $\omega, \omega' \in \Omega$, let $A_{\omega,\omega'}$ be an element of $\mathcal{F}$ that contains ω but not ω' (if there is such an element) and $A_{\omega,\omega'} = \Omega$ otherwise. Let $A_\omega = \cap\{A_{\omega,\omega'} : \omega' \in \Omega\}$. Begin by showing that $A_\omega \in \mathcal{F}$ and that if $\omega \in A \in \mathcal{F}$ then $A_\omega \subset A$.)