Real Analysis Exam Part II-Department of Mathematics, Florida State University

Jan. 5, 2008

Work three of the following four problems. Clearly indicate which three you have worked.

4. Suppose that f is nonnegative on a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$. Show that

$$E = \{(\omega, y) \in \Omega \times \mathbb{R} | 0 \leq y \leq f(\omega)\}$$

is measurable and ~~it equals~~

$$\int_\Omega f d\mu = (\mu \times \lambda)(E).$$

Here λ is Lebesgue measure.

5. Suppose that μ, ν, ρ are σ-finite measures on $(\Omega, \mathcal{F})$. Assume Radon-Nikodym derivatives are nonnegative and finite. Show that $\nu << \mu$ and $\mu << \rho$ implies that $\nu << \rho$ and

$$\frac{d\nu}{d\rho} = \frac{d\nu}{d\mu}\frac{d\mu}{d\rho}.$$

6. Suppose that $Y_1, Y_2, \ldots$ are independent positive random variables with $E(Y_n) = 1$ for each n. Let $X_n = Y_1 \cdots Y_n$. Show that $\{X_n\}$ is a martingale and converges with probability 1 to an integrable X.

7. Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be sequences of measurable sets in a measurable space $(\Omega, \mathcal{F})$. Show that $(\liminf_n A_n) \cup (\liminf_n B_n) \subset \liminf_n (A_n \cup B_n)$. Show by example that the inclusion can be strict.

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Work three of the following four problems. Clearly indicate which three you have worked.

1. Determine for which real α the limit converges :

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \int_{\frac{1}{n}}^{1} (e^{(\frac{n}{nx+1})} - e^{\frac{1}{x}})dx.$$

2. Determine for which values of $\alpha > 0$ the following integral converges :

$$\int_0^1 (\int_x^1 \frac{x^{\alpha}}{x^{\alpha}+y} dy)dx.$$

3. Suppose that $A_1, A_2, \ldots$ are independent measurable sets in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Show that

$$P(\cap_{n=1}^{\infty} A_n) = \Pi_{n=1}^{\infty} A_n$$

and

$$P(\cup_{n=1}^{\infty} A_n) = 1 - \Pi_{n=1}^{\infty}(1 - P(A_n)).$$

4. Determine for which $p > 0$ the following implication is true :
If $f \in L^p(\mathbb{R})$, then

$$\lim_{t \to 0} \int |f(x+t) - f(x)|^p dx = 0.$$