Measure and Integration Qualifying Exam – Fall 2013

Do all problems; state any theorems you use. Assume all functions are real-valued.

1. Let μ and ν be sigmafinite measures on the measurable space $(X, \mathcal{M})$. Suppose $\nu << \mu$ and let f be a nonnegative $\mathcal{M}$-measurable function on X. Prove:

$$\int_X f \, d\nu = \int_X f \left[\frac{d\nu}{d\mu}\right] \, d\mu.$$

2. Suppose $f \in L^1(\mathbf{R})$ and, for $t \in \mathbf{R}$, define the translate f_t by

$$f_t(x) = f(x - t), \quad x \in \mathbf{R}.$$

(a) For each fixed t, explain why f_t is measurable.

(b) For any $\epsilon > 0$, show there exists $\delta > 0$ such that if $|t_1 - t_2| < \delta$, then $||f_{t_1} - f_{t_2}||_1 < \epsilon$, where $|| \cdot ||_1$ denotes the norm in $L^1(\mathbf{R})$.

3. Let f be a strictly increasing function on $[0, 1]$. By Lebesgue's theorem, f is therefore differentiable almost everywhere on $(0, 1)$.

(a) Prove that f', the derivative of f, is integrable on $[0, 1]$ and

$$\int_0^1 f' \, dx \leq f(1) - f(0).$$

(b) Give an example of such a function f for which this inequality is strict.

4. Suppose $f : [0, 1] \to [0, 1]$ is Lebesgue measurable. Show that the graph Γ of f, $\Gamma = \{(x, f(x)) : x \in [0, 1]\}$, is measurable and has measure zero with respect to Lebesgue measure on $[0, 1] \times [0, 1]$.

5. Suppose $1 < p < \infty$, and $f \in L^p([0, 1])$. Prove or give a counter-example:

$$\lim_{j \to \infty} j^{(2p-2)/p} \left| \int_{1/(j+1)}^{1/j} f(x) \, dx \right| = 0.$$

Hint: Hölder.

6. For sets $A, B \subset \mathbf{R}^n$, define

$$A + B = \{x \in \mathbf{R}^n : x = a + b \text{ for some } a \in A \text{ and } b \in B\}.$$

(a) Give an example of closed sets A and B such that $A + B$ is not closed.

(b) If A and B are closed, show that $A + B$ is measurable.