Doctoral Preliminary Examination in Analysis, January 2005

1. (a) Prove that if $f : (a, b) \to I\!R$ is absolutely continuous, then it is uniformly continuous on (a, b), or give a counterexample.

   (b) Prove that if f is uniformly continuous on $[0, M)$ for each $M > 0$ and if $\lim_{x\to\infty} f(x) = L < \infty$, then f is uniformly continuous on $[0, \infty)$.

2. Determine the pairs (α, β) of $\alpha > 0$ and $\beta > 0$ for which

$$\lim_{N\to\infty} N^{-\alpha} \sum_{k-1}^{n} k^{\beta} \log k$$

   is finite. Here N is an integer > 0.

3. Suppose that L_1 and L_2 are λ-systems in a probability space. Is it true that the σ-fields that they generate are independent if and only if the π-systems they generate are independent. Give proof or counterexample.

4. Suppose that $f_n : I\!R \to I\!R$ are integrable for each $n = 1, 2, \ldots, f_n \to f$ a.e. and

$$\int f_n dm \to \int f dm$$

   as $n \to \infty$ where dm is Lebesgue measure. Does it follow that f_n converges to f in L^1? If so, give a proof, otherwise a counterexample.

5. Suppose that $f \in L^1[0, 2\pi]$. Show that

$$\lim_{n\to\infty} \int_0^{2\pi} f(t) \sin(nt)\, dt = 0.$$

   Hint: Suppose f is a step function.

6. Suppose that $g \in L^\infty(I\!R)$ is uniformly continuous and $f \in L^1(I\!R)$. Show that the convolution

$$f * g(t) = \int_{-\infty}^{\infty} f(t - s) g(s)\, ds$$

   is uniformly continuous on $I\!R$.