Doctoral Qualifying Examination **Measure and Integration** **10 January 2009**

1. Let $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ be sequences of subsets of a set Ω. Consider the possible equality

$$\limsup_n (A_n \cap B_n) = (\limsup_n A_n) \cap (\limsup_n B_n).$$

   (a) Show that equality may fail, even if $\Omega = \{1, -1\}$.

   (b) Show that equality holds (for any Ω) if $\limsup_n A_n = \lim_n A_n$.

2. Suppose $f : [0,1] \to [0,1]$ is continuous and nondecreasing and satisfies $f(0) = 0$ and $f(1) = 1$. For $t \in [0,1]$ define $\phi(t) = \lambda\{x \in [0,1] : f(x) \leq t\}$, where λ denotes Lebesgue measure. Carefully consider the following:

   (a) Is ϕ necessarily continuous from the left?

   (b) Is ϕ necessarily continuous from the right?

   (c) Do the answers to (a) and/or (b) change if f is strictly increasing?
   Note: an answer without proof will receive no credit.

3. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in $[0,1]$, and let

$$A = [0,1] \cap (\bigcup_{n=1}^{\infty} (q_n - \frac{1}{2^{n+2}}, q_n + \frac{1}{2^{n+2}}))$$

   Let $f = \mathbb{I}_A : [0,1] \to \mathbb{R}$, and let λ denote Lebesgue measure on the Borel subsets of $[0,1]$.

   (a) Show that $\int_{[0,1]} f \, d\lambda \leq \frac{1}{2}$.

   (b) Suppose that $g : [0,1] \to \mathbb{R}$ is Riemann integrable. Show that $\int_{[0,1]} |f - g| \, d\lambda > 0$. (Hint: Show that if $\int_{[0,1]} |f - g| \, d\lambda = 0$ then every upper sum for g will be at least 1.)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, and let λ denote Lebesgue measure. Show that

$$\frac{1}{8} \left| \int_{[-8,8]} f \, d\lambda \right| \leq \left(\int_{\mathbb{R}} f^4 \, d\lambda \right)^{\frac{1}{4}}.$$

5. Let

$$f_n(x) = \frac{\sin((x + \frac{1}{n})^2) - \sin((x - \frac{1}{n})^2)}{\sin(\frac{1}{n})}.$$

   Find (if it exists) $\lim_{n\to\infty} \int_{[0,1]} f_n \, d\lambda$, where λ denotes Lebesgue measure. Be sure to verify the hypotheses of any theorems from measure theory that you may use.

6. Let $\{X_n, \mathcal{F}_n\}$ be a nonnegative martingale defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Show that there exists an integrable random variable $X : \Omega \to \mathbb{R}$ so that $X_n \to X$ almost surely.

   (b) Show that if $\{\mathbb{E}[X_n^2]\}$ is bounded then, for all n, $\mathbb{E}[X|\mathcal{F}_n] = X_n$ almost surely.