Qualifying test

Solve three problems from Part 1 and three problems from Part 2

In what follows, m_p denotes the Lebesgue measure on $\mathbb{R}^p$.

Part 1

Problem 1: Assume $(X, \mathfrak{A}, \mu)$ be a measure space, and the measurable sets A_n are such that

$$\mu(A_n) < n^{-5/4}, \quad n = 1, 2, \ldots.$$

Show that

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j\right) = 0.$$

Problem 2: Let $(X, \mathfrak{A}, \mu)$ be a σ-finite measure space. Show that there exists a finite measure ν on $(X, \mathfrak{A})$ such that $\mu \ll \nu$ and $\nu \ll \mu$.

Problem 3: Assume f is a continuous function from $[0,1]$ to $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_0^1 (n+1)x^n f(x)dx = f(1).$$

Hint: make a change of variables.

Problem 4: a. Assume f is absolutely continuous on $[0,1]$, and $f'(x) \geqslant 0$ almost everywhere on $[0,1]$ (with respect to m_1). Is it true that $f(0) \leqslant f(1)$?

b. Assume f is continuous on $[0,1]$ and for some positive δ we have $f'(x) \geqslant \delta$ almost everywhere on $[0,1]$. Is it true that $f(0) \leqslant f(1)$?

Part 2

Problem 5: Assume $f\colon \mathbb{R} \to \mathbb{R}$ is a non-negative summable function (with respect to m_1). Show that

$$\int\limits_{(0,\infty)} m_1\left(\{x\colon f(x) \geqslant y\}\right) = \int\limits_{\mathbb{R}} \sin(f(x))dm_1(x).$$

Hint: use Fubini's theorem, but justify every step.

Problem 6: Assume

$$\int\limits_{(1,\infty)} |f|^3 dm_1 < \infty.$$

Is it true that

$$\int_{(1,\infty)} |f|^5 dm_1 < \infty?$$

Problem 7: Assume X, Y, Z are Banach spaces, and let $T\colon X \to Y$ and $S\colon Y \to Z$ be linear operators. Assume further that S is continuous and injective, and $S \circ T$ is continuous (from X to Z). Prove that T is continuous.

Hint: Closed graph theorem.

Problem 8: Let $p \in (1, \infty)$ and q be such that $1/p + 1/q = 1$. Assume $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $L^p(\mathbb{R})$ such that for every $g \in L^q(\mathbb{R})$ we have

$$\sup_n \int |f_n(x)g(x)| dm_1(x) < \infty.$$

Show that

$$\sup_n \|f_n\|_p < \infty.$$

Hint: this is based on another consequence of the Baire category theorem.