Qualifying test

Solve any five problems

In what follows, m_p denotes the Lebesgue measure on $\mathbb{R}^p$.

Problem 1: Let A, B, C be three Lebesgue measurable sets in $[0,1]$. Assume that almost every point x in $[0,1]$ belongs to at least two of these sets. Prove that at least one of these sets has Lebesgue measure greater or equal to $2/3$.

Problem 2: Construct a sequence of functions $\{f_n\}_{n=1}^{\infty}$, and a function g such that:

$$\int_{\mathbb{R}} |f_n| dm_1 = 2, \quad \int_{\mathbb{R}} |g| dm_1 = 1,$$

and $f_n \to g$ almost everywhere.

Problem 3: Assume $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-decreasing absolutely continuous functions on $[0,1]$. Assume also that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges to a real number for every x. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is absolutely continuous on $[0,1]$.

Problem 4: Assume f is a continuous bounded function on $[0,\infty)$. Show that

$$\lim_{n\to\infty} \int_{[0,\infty)} n e^{-nx} f(x) dm_1(x) = f(0).$$

Problem 5: a. Prove that $L^3(0,1) \subset L^2(0,1)$.

b. Give an example of a bounded linear functional F on $L^3(0,1)$ which is not a restriction of a bounded linear functional on $L^2(0,1)$. I.e., there is no linear functional G on $L^2(0,1)$ such that

$$F(f) = G(f), \quad \forall f \in L^3(0,1).$$

Problem 6: Assume X is a Banach space, and $\{x_n\}_{n=1}^{\infty} \subset X$. Show that the sequence $\{x_n\}$ is bounded in X if and only if for every bounded linear functional f, the sequence $\{f(x_n)\}$ is bounded in $\mathbb{R}$.

Problem 7: Show that $\ell^1(\mathbb{N})$ is not a Hilbert space; i.e., there is no inner product $(\cdot,\cdot)$ such that

$$(x,x) = \|x\|_1^2, \quad \forall x \in \ell^1(\mathbb{N}).$$

Hint: $\ell^1(\mathbb{N})$ is separable.