Qualyfying test

In all problems, m_1 denotes the Lebesgue measure on $\mathbb{R}$ defined on the Lebesgue sigma-algebra $\mathfrak{M}_1$.

Problem 1: Let f be a summable function on $\mathbb{R}$ with respect to the Lebesgue measure. Prove that the following are equivalent.

- $f = 0$ a.e. on $\mathbb{R}$;
- $\int_{\mathbb{R}} fg dm_1 = 0$ for every bounded measurable function g;
- $\int_A f dm_1 = 0$ for every measurable set A;
- $\int_G f dm_1 = 0$ for every open set G.

Problem 2: Let $\{f_n\}$ be a sequence of non-negative summable functions on the real line with respect to m_1, and for every x assume $f_{n+1}(x) \leqslant f_n(x)$. Prove that if

$$\int_{\mathbb{R}} f_n dm_1 \to 0 \text{ as } n \to \infty,$$

then $f_n \to 0$ almost everywhere. Is the converse true?

Problem 3: a) Show that if f is a non-negative summable function on $[0,1]$ with respect to m_1, then $f(x) < \infty$ for almost every x.

b) Let $\{q_n\}_{n=1}^{\infty} = \mathbb{Q} \cap [0,1]$, and

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{|x - q_n|}}.$$

Show that f is unbounded on any open interval $I \subset [0,1]$. Is it true that $f(x) < \infty$ for almost every x (explain)?

Problem 4: Let

$$f(x,y) = \begin{cases} 1/x^2, & 0 < y < x < 1 \\ -1/y^2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Compute

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) dm_1(x) \right) dm_1(y), \text{ and } \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) dm_1(y) \right) dm_1(x).$$

Explain why the results are consistent with the Fubini theorem.