Measure and Integration Qualifying Exam DRAFT – Fall 2017

Do all problems; state any theorems you use.

Notation: $(\mathbb{R}^d, \mathcal{L}, m)$ denotes the usual d-dimensional Lebesgue measure space.

1. Suppose a function $f : [0,1] \to \mathbb{R}$ is differentiable almost everywhere, and for some $\epsilon > 0$ we have $f'(x) > \epsilon$ for almost all x. Is it true that f must be (not necessarily strictly) increasing? Justify.

2. For any set $E \subset \mathbb{R}^d$ and $y \in \mathbb{R}^d$, define the y-translate of E by $E + y = \{x + y \in \mathbb{R}^d : x \in E\}$.

(a) If $E \in \mathcal{L}$, prove that $E + y \in \mathcal{L}$ for any $y \in \mathbb{R}^d$, and that $m(E) = m(E + y)$.

(b) Fix $y \in \mathbb{R}^d$ and let $f : \mathbb{R}^d \to [0, +\infty]$ be Lebesgue measurable. Define $g(x) = f(x + y)$ for all $x \in \mathbb{R}^d$. Prove that g is Lebesgue measurable and

$$\int_{\mathbb{R}^d} g\, dm = \int_{\mathbb{R}^d} f\, dm.$$

3. Let f_n be a sequence of real-valued measurable functions on a measurable space $(X, \mathcal{M})$.

(a) Prove that $E = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

(b) Prove that the function $f : E \to \mathbb{R}$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is a measurable function.

4. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and $f : X \to [0, +\infty]$ a measurable function. Show that

$$\int_X f(x)\, d\mu(x) = \int_{[0,+\infty]} \mu(\{x \in X : f(x) \geq \lambda\})\, dm(\lambda).$$

5. Suppose that K is a compact set in $\mathbb{R}^d$ and m denotes Lebsgue measure. For $\delta > 0$, define the δ-neighborhood K_δ of K by

$$K_\delta = \{k + x : k \in K, |x| < \delta\}.$$

(a) Show that $\lim_{\delta\to 0^+} m(K_\delta) = m(K)$

(b) Show that the statement in (a) may fail if one assumes only that K is a Borel set.

6. Let $g : \mathbb{R}^d \to \mathbb{R}$ be non-negative and Lebesgue measurable. For any Lebesgue measurable set E, Define

$$\mu(E) = \int_{\mathbb{R}^d} g 1_E\, dm.$$

(a) Show that μ is a countably additive measure on $(\mathbb{R}^d, \mathcal{L})$.

(b) Show that $\int f\, d\mu = \int fg\, dm$ for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$.