PDE 2, August 2018 Qualifying Exam

1. Consider the variable-coefficient PDE

$$y u_{xx} - 2u_{xy} + x u_{yy} = 0 \tag{5}$$

Find the regions in the xy plane where this PDE is elliptic, parabolic, and hyperbolic. Sketch these regions.

2. Consider the Hopf equation

$$u_t + u u_x = 0 \qquad \text{for } -\infty < x < \infty,\ t > 0 \tag{6}$$
$$u(x,0) = \phi(x) \qquad \text{for } -\infty < x < \infty \tag{7}$$

where $\phi(x)$ has compact support (i.e. there exists some finite interval such that ϕ vanishes outside that interval). Define the 'mass' of the solution as

$$M(t) = \int_{-\infty}^{\infty} u(x,t)dx \tag{8}$$

a) Prove that $M(t)$ is conserved.
(Hint: the conservative form of the PDE might help, but this is not the only approach that works.)

b) Now consider Burgers equation

$$u_t + u u_x = \nu u_{xx} \qquad \text{for } -\infty < x < \infty,\ t > 0 \tag{9}$$
$$u(x,0) = \phi(x) \qquad \text{for } -\infty < x < \infty \tag{10}$$

where again $\phi(x)$ has compact support. Is the mass still conserved? If so, prove it. If not, calculate the rate at which mass changes.

3. Consider the wave equation in arbitrary spatial dimension $\mathbb{R}^d$

$$u_{tt} - c^2 \Delta u = 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^d, t > 0 \tag{11}$$
$$u(\boldsymbol{x}, 0) = \phi(\boldsymbol{x}) \tag{12}$$
$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}) \tag{13}$$

where ϕ and ψ both have finite L^1 and L^2 norms. We want to prove that a solution exists using the Fourier transform

$$f(\boldsymbol{x}, t) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{k}, t) e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k} \tag{14}$$
$$\hat{f}(\boldsymbol{k}, t) = \int_{\mathbb{R}^d} f(\boldsymbol{x}, t) e^{-2\pi i \boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} \tag{15}$$

To this end, do the following:
a) Find the solution to the PDE in Fourier space, i.e. find the function $\hat{u}(\boldsymbol{k}, t)$.

b) Given that the transforms of ϕ and ψ both have finite L^1 and L^2 norms, show that $\hat{u}(\boldsymbol{k}, t)$ also has finite L^1 and L^2 norms.

c) Why does this imply that a physical solution $u(\boldsymbol{x}, t)$ exists for all time?

4. Consider the nonlinear PDE for $u(x, t)$

$$u_t = \sin(au) + Du_{xx}\,, \qquad \text{for } 0 < x < L,\ t > 0 \tag{16}$$
$$u(0, t) = u(L, t) = 0\,, \qquad \text{for } \ t > 0 \tag{17}$$

where a and D are constants and $D > 0$.

a) Perform linear stability analysis about the trivial solution $u_0(x, t) = 0$. What is the condition for an instability to exist?

b) Now let $L = \pi$, $D = 2$, and $a = 10$. In this case, how many unstable modes are there? Which one will dominate in long time?