PDE I, August 2018 Qualifying Exam

1. Consider the driven heat equation for $u(x,t)$

$$u_t - ku_{xx} = 1 \qquad \text{for } 0 < x < L,\ t > 0 \tag{1}$$

with trivial initial condition $u(x,0) = 0$.

a) For vanishing Dirichlet boundary conditions (at $x = 0$ and L), find the exact solution. Plot the solution for a few different time values.

b) For vanishing Neumann boundary conditions (at $x = 0$ and L), find the exact solution and again plot it.

c) In both cases, make sure your plots are clear and that they clearly indicate the long-time behavior. Is the long-time behavior different in the two cases?

2. Let $\Omega \subset \mathbb{R}^n$ be an open set, and consider a suitable space of real-valued functions $u : \Omega \to \mathbb{R}$ equipped with the standard inner product

$$\langle u, v \rangle = \int_\Omega u\, v\, dV \tag{2}$$

Let $\mathscr{L}$ be a self-adjoint linear operator on this function space. Prove that eigenfunctions belonging to distinct eigenvalues of $\mathscr{L}$ are orthogonal with respect to the inner product.

3. For each case below, determine if the given function space is a vector space over $\mathbb{R}$. If it is a vector space, give a brief proof. If it is not, specify an axiom that fails to hold by giving a concrete counterexample.

a) $\mathcal{F}_1 = \{f : (0, 10) \to \mathbb{R} \text{ such that } f(4) = 0\}$

b) $\mathcal{F}_2 = \{f : (0, 10) \to \mathbb{R} \text{ such that } f(4) = 2\pi\}$

c) $\mathcal{F}_3 = \{f : (0, 10) \to \mathbb{R} \text{ such that } f(x) \geq 0 \ \forall x\}$

d) $\mathcal{F}_4 = \{f : (0, 10) \to \mathbb{R} \text{ such that } f'(x) \text{ exists } \forall x \text{ and } f'(4) = 0\}$

4. Consider the Poisson problem with Dirichlet boundary conditions in a bounded, simply connected, open domain $\Omega \subset \mathbb{R}^2$ (i.e. a 'nice' domain),

$$\Delta u = f \quad \text{for } \boldsymbol{x} \in \Omega \tag{3}$$

$$u = g \quad \text{for } \boldsymbol{x} \in \partial\Omega \tag{4}$$

Prove that if a solution exists, then it is unique. You may use the maximum principle for harmonic functions.