PDE 1, August 2017 Qualifying Exam

1. Consider the forced heat equation for $u(x,t)$

$$u_t - ku_{xx} = g(x) \qquad \text{for } 0 < x < L \tag{1}$$

with vanishing Neumann boundary conditions. Let $g(x) = A\cos(\pi x/L)$ and consider vanishing initial condition $u(x,0) = 0$.

a) Find the exact solution via separation of variables.

b) Let $A = 1$ and $L = \pi$. Plot the solution at a few time values. Clearly indicate what the solution looks like at $t = 0$ and as $t \to \infty$.

2. Consider the Poisson problem with Dirichlet boundary conditions in a bounded, simply connected, open domain $\Omega \subset \mathbb{R}^2$ (i.e. a 'nice' domain),

$$\Delta u = f \qquad \text{for } \boldsymbol{x} \in \Omega \tag{2}$$

$$u = g \qquad \text{for } \boldsymbol{x} \in \partial\Omega \tag{3}$$

Prove that if a solution exists, then it is unique. You may use the maximum principle for harmonic functions.

3. Consider the space of functions $\mathcal{F} = \{f : (0,1) \to \mathbb{R},\ \text{such that } f \text{ is continuous}\}$.

a) Find a sequence of functions in $\mathcal{F}$ that converges to $f = 0$ (the trivial function) in L^2 **but not** uniformly.

b) Find a sequence of functions in $\mathcal{F}$ that converges to the trivial function pointwise **but not** in L^2.

4. For each function $f(x)$ below, defined on the interval $x \in [-\pi, \pi)$, consider the periodic extension $f_{ext}(x) : \mathbb{R} \to \mathbb{R}$.

i) $f(x) = x$

ii) $f(x) = \pi^2 - x^2$

iii) $f(x) = (\pi^2 - x^2)^2$

a) In each case, sketch $f_{ext}(x)$ and determine the largest non-negative integer n such that $f_{ext} \in C^n$.

b) Based on this information, can you get an upper bound on the decay rate of the Fourier coefficients in each case?

PDE 2, August 2017 Qualifying Exam

1. Consider the variable-coefficient PDE for $u(x,y)$

$$(1+x)\,u_{xx} + 2xy\,u_{xy} - y^2\,u_{yy} = 0 \tag{4}$$

a) Find the regions in the xy plane where this PDE is elliptic, parabolic, and hyperbolic.

b) For this PDE, would you expect information to propagate at a finite or infinite rate?

2. Consider the Laplace's equation with Dirichlet boundary conditions in the upper half space, $\Omega = \{(x,y,z) : z > 0\}$

$$\Delta u = 0 \qquad \text{in } \Omega \tag{5}$$

$$u = g(x,y) \qquad \text{on } \partial\Omega \tag{6}$$

Using 3D Green's functions, find the solution to this BVP through the representation formula. Make sure your final result is as explicit as possible.

3. Consider the Hopf equation for $u(x,t)$

$$u_t + uu_x = 0 \qquad \text{for } x \in (-\infty,\infty),\ t>0 \tag{7}$$
$$u(x,0) = e^x - x \qquad \text{for } x \in (-\infty,\infty) \tag{8}$$

a) Take the method of characteristics as far as possible in determining the solution. Sketch the characteristics and make a qualitative sketch of the solution at a few time values.

b) Explicitly find the solution at $t=1$ and sketch it. What is the domain at of $u(x,t)$ at $t=1$? Reconcile this with your qualitative sketches from part a.

4. The goal of this problem is to prove that the reverse-time heat equation is unstable with respect to initial data. Consider the IBVP for $u(x,t)$

$$u_t + ku_{xx} = 0 \qquad \text{for } 0<x<L,\ t>0 \tag{9}$$
$$u(0,t) = 0 \tag{10}$$
$$u(L,t) = 0 \tag{11}$$
$$u(x,0) = 0 \qquad \text{for } 0<x<L \tag{12}$$

a) Find the general solution that satisfies the PDE and boundary conditions (but not yet the initial condition).

b) Of course, considering the initial condition produces the trivial solution. Now show that this solution is unstable by explicitly constructing a solution $\tilde{u}(x,t)$ with small initial data, but that grows unbounded in time. Given a $\delta>0$, explicitly construct some initial data $\tilde{u}_0(x)$ such that $|\tilde{u}_0(x)|<\delta$, but at $t=1$, we have $\max_{x\in(0,L)}|\tilde{u}(x,1)| \geq 1$.