PDE 1, January 2018 Qualifying Exam

1. Consider a function $f : (-\pi, \pi) \to \mathbb{R}$. Prove that if f has a Fourier series, then the Fourier series is unique.

2. Consider the *exterior* of the unit disk,

$$\Omega = \{\boldsymbol{x} = (x, y) \text{ such that } x^2 + y^2 > 1\}$$

Using polar coordinates, (r, θ), the domain Ω is described simply by $r > 1$ and its boundary $\partial\Omega$ by $r = 1$.

Consider a Dirichlet problem on this domain,

$$\Delta u = 0 \qquad \text{for } r > 1 \tag{1}$$

$$u = \sin^2\theta \qquad \text{for } r = 1 \tag{2}$$

with the far-field condition,

$$|u| < M \qquad \text{as } r \to \infty \tag{3}$$

for some $M \in \mathbb{R}$. That is, u is bounded in the far-field.
a) Find the general solution that satisfies the PDE and far-field condition (do not yet insert the boundary conditions).
b) Insert the boundary conditions to find the exact solution to the BVP. You may use the trigonometric identity $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$.
c) Using your solution, calculate $\lim_{r\to\infty} u$.

3. Consider the so-called Klein-Gordon PDE

$$u_{tt} - c^2 \Delta u + m^2 u = 0 \tag{4}$$

in the unit square $\Omega \subset \mathbb{R}^2$, $\Omega = \{(x, y) \text{ such that } x \in (0,1) \text{ and } y \in (0,1)\}$, with vanishing Dirichlet boundary conditions. Find the general solution using eigenfunction expansion (i.e. separation of variables).

Can you draw any conclusions about the orthogonality of eigenfunctions? If so, why? Are there any repeated eigenvalues in this problem? If so, are the associated eigenfunctions orthogonal?

4. Consider the Poisson problem with Dirichlet boundary conditions in a bounded, simply connected, open domain $\Omega \subset \mathbb{R}^2$ (i.e. a 'nice' domain),

$$\Delta u = f \quad \text{for } \boldsymbol{x} \in \Omega \tag{5}$$

$$u = g \quad \text{for } \boldsymbol{x} \in \partial\Omega \tag{6}$$

Prove that if a solution exists, then it is unique. You may use the maximum principle for harmonic functions.

PDE 2, January 2018 Qualifying Exam

1. Consider the variable-coefficient PDE

$$y u_{xx} - 2u_{xy} + x u_{yy} = 0 \tag{7}$$

Find the regions in the xy plane where this PDE is elliptic, parabolic, and hyperbolic. Sketch these regions.

2. Consider the Hopf equation for $u(x,t)$

$$u_t + u u_x = 0 \qquad \text{for } x \in (-\infty, \infty), t > 0 \tag{8}$$

$$u(x,0) = \sin(x) \qquad \text{for } x \in (-\infty, \infty) \tag{9}$$

a) Apply the method of characteristics to: (i) determine the characteristic map, (ii) sketch the characteristics in the x-t plane, (iii) sketch the solution $u(x,t)$ for a few time values.

b) Do shocks form in this problem? If so, estimate the time at which shocks first form (either exactly or a bound).

c) Sketch the long-time behavior of the solution (i.e. the solution that satisfies the associated conservation law).

3. Consider the 1D free-space wave equation for $u(x,t)$.

$$u_{tt} - c^2 u_{xx} = 0 \qquad \text{for } x \in (-\infty, \infty), t > 0 \tag{10}$$
$$u(x,0) = \phi(x) \qquad \text{for } x \in (-\infty, \infty) \tag{11}$$
$$u_t(x,0) = \psi(x) \qquad \text{for } x \in (-\infty, \infty) \tag{12}$$

Prove uniqueness of the solution to this IVP (you do not have to prove existence).

4. Consider the Laplace's equation with Neumann boundary conditions in the upper half space, $\Omega = \{(x,y,z) : z > 0\}$

$$\Delta u = 0 \qquad \text{in } \Omega \tag{13}$$
$$\frac{\partial u}{\partial n} = g(x,y) \quad \text{on } \partial\Omega \tag{14}$$

Using 3D Green's functions, find the solution to this BVP through the representation formula. Make sure your final result is as explicit as possible.