PDE I, August 2019 Qualifying Exam

1. Consider the 1D heat equation for the temperature $u(x,t)$ in a narrow rod of length L. You are heating the left end ($x = 0$) with a constant heat flux J_0. The right end ($x = L$) is perfectly insulated. Find an exact solution to the PDE that gives the correct long-time behavior of $u(x,t)$, no matter what the initial conditions are. Sketch a graph of the solution.

2. Consider the 1D heat equation for $u(x,t)$ with vanishing Dirichlet conditions

$$u_t = ku_{xx} \qquad \text{for } -\pi < x < \pi,\ t > 0 \tag{1}$$

$$u(-\pi,t) = u(\pi,t) = 0 \tag{2}$$

Consider a piecewise initial condition

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases} \tag{3}$$

a) Find the solution using Fourier series. (Note: even thought the initial condition does not satisfy the boundary conditions, it is still possible to find a solution.)

b) Using the Fourier decay theorems, prove that the solution you found is infinitely smooth for any positive time, $t > 0$, no matter how small.

3. Let $\Omega \subset \mathbb{R}^n$ be an open set, and consider a suitable space of complex-valued functions $u : \Omega \to \mathbb{C}$ equipped with the standard inner product

$$\langle u, v \rangle = \int_{\Omega} u \, \bar{v} \, dV \tag{4}$$

Let $\mathscr{L}$ be a self-adjoint linear operator on this function space. Prove the following:

a) All eigenvalues of $\mathscr{L}$ are real.

b) Eigenfunctions belonging to distinct eigenvalues of $\mathscr{L}$ are orthogonal with respect to the inner product.

4. Consider a domain D which can be bounded or unbounded. In parts a and b, consider a sequence of continuous functions $f_n : D \to \mathbb{R}$ and a continuous function $f : D \to \mathbb{R}$.

a) Consider the domain $D = (0, 10)$. Prove or disprove: if $f_n \to f$ uniformly then $f_n \to f$ in L^2.

b) Consider the domain $D = (0, \infty)$. Prove or disprove: if $f_n \to f$ uniformly then $f_n \to f$ in L^2.