PDE 2, August 2019 Qualifying Exam

1. Consider the IBVP for $u(x,t)$

$$u_t = Du_{xx} \qquad \text{for } 0 < x < 2\pi, t > 0 \tag{5}$$
$$u(x,0) = \sin(x) \qquad \text{for } 0 < x < 2\pi \tag{6}$$
$$u(0,t) = 0 \qquad \text{for } t > 0 \tag{7}$$
$$u(2\pi,t) = 2\sinh(t) \qquad \text{for } t > 0 \tag{8}$$

where $D > 0$. Using the maximum principle, decide if each of the following are possible or impossible, and give a brief justification for your answer. In each case, suppose x_0 is an arbitrary point inside the interval $(0, 2\pi)$.
a) $u(x_0, 1) = -3/2$
b) $u(x_0, 1) = -1/2$
c) $u(x_0, 1) = 3/2$
d) $u(x_0, 1) = 3$
e) $\lim_{t\to\infty} u(x_0, t) = +\infty$
f) $\lim_{t\to\infty} u(x_0, t) = 5$

2. Let $f : \mathbb{R} \to \mathbb{R}$ with $f \in L^1$. Define the distribution f as

$$(f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\, dx \tag{9}$$

for any function ϕ in the space of C^∞ and compact support. Prove that this distribution is continuous. i.e. Prove that if $\phi_n \to \phi$ uniformly, then $(f, \phi_n) \to (f, \phi)$.

3. Consider Laplace's equation on the interior of the unit disk Ω, with Neumann boundary conditions

$$\Delta u = 0 \qquad \text{for } r < 1 \tag{10}$$
$$\frac{\partial u}{\partial n} = g(\theta) \quad \text{for } r = 1\,. \tag{11}$$

Suppose $g(\theta)$ is nice enough to have a Fourier series,

$$g(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta) \tag{12}$$

a) Find the exact solution for $u(x,t)$. Write your answer in terms of the coefficients a_n and b_n given above.

b) State the compatibility condition for the Poisson equation with Neumann boundary conditions.

c) Appealing to the solution you found in part a, demonstrate that a solution $u(x,t)$ exists **if and only if** the compatibility condition is satisfied.

4. Consider the *inhomogeneous* Burger-Hopf equation for $u(x,t)$

$$u_t + uu_x = 2\,, \qquad x \in (-\infty,\infty), t > 0 \tag{13}$$
$$u(x,0) = \phi(x) \tag{14}$$

where $\phi(x)$ is some generic bell curve shape.

a) Apply the method of characteristics to this problem. Determine u on each characteristic curve.

b) Sketch the characteristics.

c) Sketch the corresponding solution at a few different time values.

d) Find the time at which a shock first develops (your answer will depend on $\phi(x)$).