**TOPOLOGY QUALIFYING EXAM August 2007**

**PART I.**

**Do one of the following:**

1.(a) Show that a space $X$ is not connected if and only if there exists a continous surjection $\phi : X \to \{0, 1\}$ to the two-point discrete space $\{0,1\}$ .

(b) Let $f : X \to Y$ be a continuous map. Show: If $X$ is connected then the image $f(X)$ is connected.

2. Show that a compact Hausdorff space is normal.

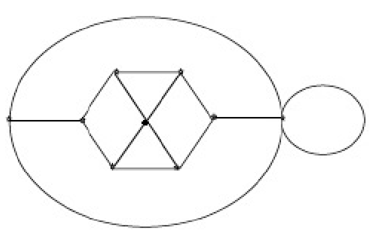
(Hint: First show that a compact Hausdorff space is regular).

**PART II.**

**Do any three of Problems 1, 2, 3, or 4 and one of Problems 5 or 6.**

1.(a) Let $\Gamma$ be a graph (i.e. a 1-dimensional CW-complex). Describe an algorithm for computing a presentation for the fundamental group of $\Gamma$ .

(b) Use the algorithm in (a) to find a presentation for the fundamental group of the graph below. (Explain how the generators relate to the graph).

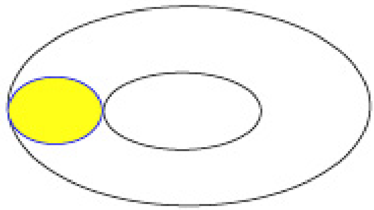


(c) Prove that any subgroup of a free group is free.

2. (a) Construct all connected covering spaces of $S^1 \vee S^2$ (the one-point union of the 1- and 2-sphere). Carefully explain why your list is complete. Describe (by pictures) the action of the group of Deck-transformations on the covering space.

(b) Construct all connected covering spaces of $P^2 \vee S^2$ (the one-point union of the projective plane and 2-sphere). Carefully explain why your list is complete. Describe (by pictures) the action of the group of Deck-transformations on the covering space.

3. Let $X$ be the union of the torus $T = S^1 \times S^1$ with the disk $D^2$, where $D^2$ is attached to $T$ by identifying $\partial D^2$ with a curve $S^1 \times \{x_0\}$, where $\{x_0\}$ is a point of $S^1$, see the figure below.



(a) Calculate $\pi_1(X)$.

(b) Construct all connected covering spaces of $X$. (Explain using pictures).

(c) Is $T$ a retract of $X$ ? (Justify your answer)

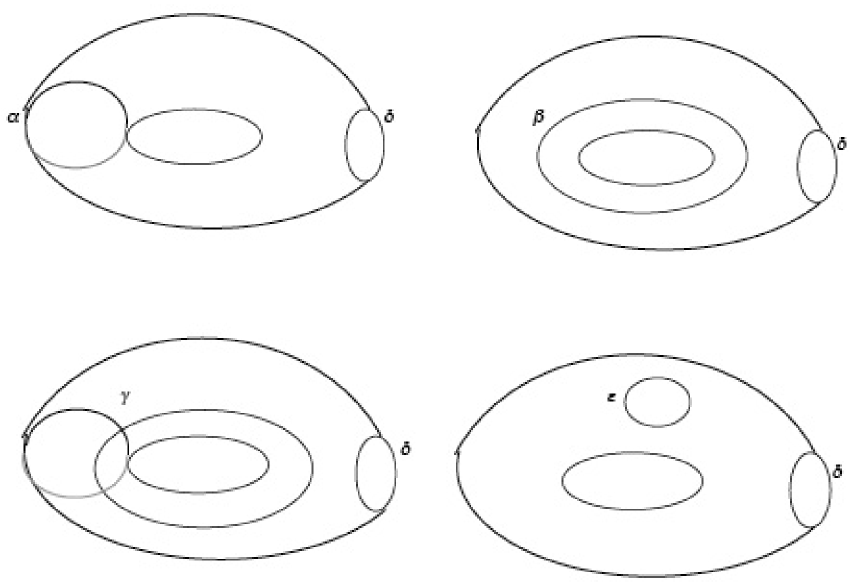
4. Let $X$ be the the torus $T = S^1 \times S^1$ with an open disk removed and let $\delta$ be the boundary curve of $X$ (= boundary of the deleted disk). Let $\alpha = S^1 \times \{x_0\}$ and $\beta = \{y_0\} \times S^1$ (for a point $\{x_0\}$ resp. $\{y_0\}$ of $S^1$), let $\epsilon$ be a circle on $X$ that bounds a disk on $X$, and let $\gamma = \alpha \cup \beta$ (a figure eight), as indicated in the figures below.

(a) Is $\gamma$ a retract of $X$ ? (Justify your answer)

(b) Is $\alpha$ a retract of $X$ ? (Justify your answer)

(c) Is $\epsilon$ a retract of $X$ ? (Justify your answer)

(d) Is $\delta$ a retract of $X$ ? (Justify your answer)



5. For a closed orientable 2-manifold $M$ of genus $g \geq 0$ compute $H_q(M)$.

(Hint: Use the the Mayer-Vietoris sequence and the fact that there is a band-neighborhood $U$ of a wedge $\Gamma$ of circles in $M$ that deformation-retracts to $\Gamma$ so that $V = \overline{M - U}$ is a disk.)

6. Let $N$ be a subset of $S^3$ that is homeomorphic to a solid torus $D^2 \times S^1$. Compute $H_q\overline{(S^3 - N)}$.

(Hint: Apply the Mayer-Vietoris sequence to $\overline{S^3 - N}$ and $N$).

**PART III.**

**Do any two of the following Problems.**

1. (a) Given a smooth map $f : M \to N$, where $M$ and $N$ are smooth oriented compact manifolds without boundary, both of the same dimension, write the definition of the *degree* of $f$ using the basic notions of differential topology. (Hint: the degree of $f$ is an integer.)

(b) Let $C$ denote the complex numbers, and let $\hat{C}$ denote the Riemann sphere, i.e. $\hat{C} = C \cup \{\infty\}$. As you know, $\hat{C}$ is a 2-dimensional manifold, diffeomorphic to the sphere $S^2$. Define $f : \hat{C} \to \hat{C}$ by

$$f(z) = \frac{z^2}{z^3 + z^2 - 1}.$$

(Of course, this formula is not defined at a few points of $\hat{C}$, but there is a unique smooth extension to all of $\hat{C}$.) Using the definition you gave in (a), calculate the degree of $f$.

2. Let $v$ be a smooth vector field on the two-dimensional sphere $S^2$. Suppose $v$ vanishes at precisely two points, $p$ and $q$, of $S^2$. If the index of $v$ at $p$ is 5, what is the index of $v$ at $q$? Be sure to explain your reasoning.

3. Let $S^4$ denote the four-dimensional sphere,
$S^4 = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 : x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1\}$.
Suppose $f : S^4 \to S^4$ is a smooth map with no fixed points.

(a) What is the degree of $f$?

(b) Is $f$ homotopic to the identity map?