TOPOLOGY QUALIFYING EXAM January 2008

PART I.

Do one of the following:

1. (a) Let $p : X \to Y$ be a quotient map and let $g : X \to Z$ be a map such that g is constant on each set $p^{-1}(y)$, for $y \in Y$. Show that there is a well-defined map $f : Y \to Z$ such that $f \circ p = g$, and that f is continuous if and only if g is continuous.

(b) Let X be the unit interval and A be the subspace consisting of 0 and 1. Use (a) to show that the quotient space X/A is homeomorphic to S^1.

2. Let X be a space and Y be a compact space. Suppose U is an open set of $X \times Y$ (with the product topology) containing the subspace $x_0 \times Y$ of $X \times Y$ (for a fixed $x_0 \in X$). Show that there is an open neighborhood W of x_0 in X such that $W \times Y \subset U$.

PART II.

Do any four of the following.

1. Let X be a path-connected space and A be a path-connected subspace containing the basepoint x_0. Let $i : A \to X$ be the inclusion map.

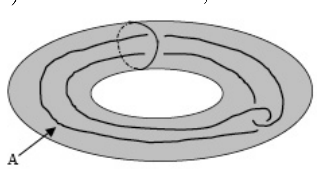
Show: $i_* : \pi(A, x_0) \to \pi(X, x_0)$ is surjective if and only if every path in X with endpoints in A is path-homotopic to a path in A.

2. Determine whether there are retractions $r : X \to A$ in the following cases (Justify your answers):

(a) $X = D^2 \vee S^1$ and $A = S^1 \vee S^1$.

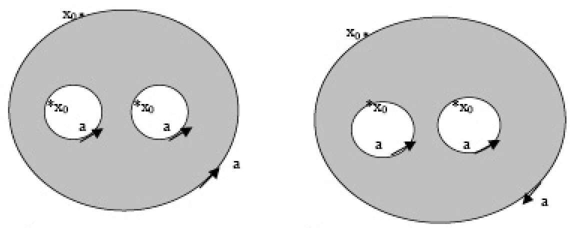
(b) X is a disk with two holes and A is one of its boundary circles.

(c) $X = S^1 \times D^2$, A is the circle shown in the figure



3. Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the connected, locally path-connected spaces X and Y.
Show that if X and Y are homotopy equivalent then $\tilde{X}$ and $\tilde{Y}$ are homotopy equivalent.

4. Let X, Y be the spaces obtained from a disk D with two holes by identifying the boundary circles a as in Fig 1, Fig 2, resp., below.



X Fig 1 Y Fig 2

Consider a CW-structure on X, Y, resp., with one vertex x_0 and three 1-cells (one is the edge a, the other two are obtained by joining the outer boundary curve of D by arcs b and c in D to the inner boundary curves).

(a) Give a presentation for $\pi(X, x_0)$ and for $\pi(Y, x_0)$.

(b) Show that X and Y are not homeomorphic. (Hint: Abelianize the fundamental groups for X and Y).

5. Let $\mathbf{Z} * \mathbf{Z_2} = \langle a\,,\, b \,|\, b^2 \rangle$ be represented by $X = S^1 \vee P^2$

For the subgroup H below construct the covering space $\tilde{X}$ by sketching a good picture for $\tilde{X}$ and explaining (in your picture) how it covers X.

In each case give a group presentation for the group G of covering transformations of the covering $p : \tilde{X} \to X$ and describe (using your picture) the action of G on $\tilde{X}$.

(a) H is the smallest normal subgroup containing b.

(a) H is the smallest normal subgroup containing a.

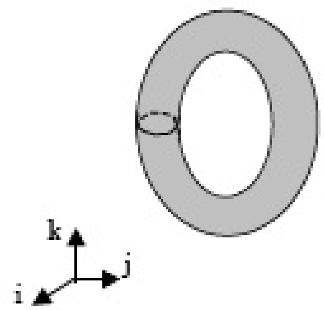
(a) H is the smallest normal subgroup containing a^2 and b.

(a) H is the trivial subgroup.

PART III.

Do any two of the following Problems.

1. Consider the torus M, oriented as shown below with relation to the standard basis for $\mathbf{R}^3$. For each $x \in M$, define $P_x : \mathbf{R}^3 \to TM_x$ to be orthogonal projection. Define a vector field $\vec{v}$ on M by $\vec{v}(x) = P_x(\vec{k})$, where $\vec{k}$ is the standard upward pointing basis vector of $\mathbf{R}^3$.



(a) Indicate on the picture the points of M at which $\vec{v}$ vanishes.

(b) Calculate the index of $\vec{v}$ at each point indicated in (a).

(c) State the Poincare-Hopf Theorem. Based on your results in (a) and (b), use this theorem to determine the Euler characteristic of M.

2. Give examples of two non-empty 0-dimensional framed submanifolds of S^2 (describe them by drawing them, together with their framings, on two separate pictures of S^2) which are not framed cobordant to each other. Prove they are not framed cobordant using material discussed in Differential Topology.

3. Consider the function $g : \mathbf{R}^2 \to \mathbf{R}$ defined by $g(x, y) = x^3 - 3xy^2 - x^2 + y^2$. Find all critical points (in $\mathbf{R}^2$) and critical values (in $\mathbf{R}$) of this function. For which elements in $y \in \mathbf{R}$ would you expect $g^{-1}(y)$ to be a submanifold of $\mathbf{R}^2$. What would be its dimension?