**TOPOLOGY QUALIFYING EXAM August 2009**

**Notations:** $X$, $Y$, ... are topological spaces, $R^n$ Euclidean $n$-space, $S^n$ unit $n$-sphere in $R^{n+1}$.

## PART I.

**Do two of the following:**

1. Suppose that f: X $\rightarrow$ Y is a homotopy equivalence.

(a) If X is path connected is Y path connected? (Give a proof or counterexample).

(b) If X is compact is Y compact? (Give a proof or counterexample).

2. Let $p : X \rightarrow Y$ be a quotient map and $h : X \rightarrow Z$ a continuous map such that for each $y \in Y$ , $h$ is constant on $p^{-1}(y)$ . Construct a continuous map $k : Y \rightarrow Z$ such that $h = k \circ p$. (Show that your map is well-defined and continuous).

3. Prove the following two statements.

(a) A compact subset of a Hausdorff space is closed.

(b) A closed subset of a compact space is compact. .

**PART II.**

**Do three of the following:**

1. (a) Let $Z$ be $\mathbf{R}^3$ with the $x-$axis and $y-$axis removed. Compute $\pi_1(Z, z_0)$ (pick a point $z_0 \in Z$). Describe curve(s) in $\mathbf{R}^3$ representing the generator(s) of $\pi_1(Z, z_0)$.

(b) Let $X$ be the quotient space of $S^2$ obtained by identifying the north and south poles to a single point. Compute $\pi_1(X, x_0)$ (pick a point $x_0 \in X$). Describe curve(s) in $X$ representing the generator(s) of $\pi_1(X, x_0)$.

(c) The Klein Bottle $K$ can be obtained from two copies $M_1$, $M_2$ of the Mobius band by gluing their boundary circles together via a homeomorphism. Use this description of $K$ to find a presentation for $\pi_1(K, y_0)$ (pick a point $y_0 \in K$). Describe curve(s) in $M_i$ representing the generator(s) of $\pi_1(K, y_0)$.

2. (a) Construct all connected covering spaces of $\mathbb{P}^2 \vee \mathbb{S}^2$ (The one-point union of the projective plane and 2-sphere).
Carefully explain why your list is complete. Describe the group of covering transformations.

(b) Construct all connected covering spaces of $\mathbb{S}^1 \vee \mathbb{S}^2$.
Carefully explain why your list is complete. Describe the group of covering transformations.

3. (a) Let $T$ be the the torus $T = S^1 \times S^1$, let $A = S^1 \times \{x_0\}$ (for a point $\{x_0\}$ of $S^1$). Let $X$ be obtained from $T$ by removing a small open disk (disjoint from $A$).

Is $A$ a retract of $X$ ? (Either describe a retraction or show that no retraction exists).

Is $A$ a deformation retract of $X$ ? (Justify your answer)

(b) Let $M$ be the Moebius band and let $C$ be its boundary curve. Is $C$ a retract of $M$? (Either describe a retraction or show that no retraction exists).

4. Let $G$ be the set of Deck transformations (covering transformations) of a covering $p : \tilde{X} \to X$.

(a) Show that $G$ is a group under composition.

(b) Let $H = p_*\pi(\tilde{X}, \tilde{x}_0) \subset \pi(X, x_0)$. Construct an epimorphism $\phi : N(H) \to G$ with kernel $H$.

(Here $N(H)$ is the normalizer of $H$ in $\pi(X, x_0)$. Define $\phi$, show it is well-defined, show $\phi$ is surjective, show $ker(\phi) = H$. You need not show that $\phi$ is a homomorphism).