TOPOLOGY QUALIFYING EXAM May 2009

Notations: X, Y, ... are topological spaces, R^n Euclidean n-space, S^n unit n-sphere in R^{n+1}, D^n unit n-ball in R^n, I unit interval in R^1.

PART I.

Do two of the following:

1.(a) Show that a space X is not connected if and only if there exists a continous surjection $\phi : X \to \{0,1\}$ to the two-point discrete space {0,1} .

(b) Let $f : X \to Y$ be a continuous map. Show: If X is connected then the image $f(X)$ is connected.

2. Prove that a compact Hausdorff space is regular.

3. (a) Construct a quotient map $p : S^1 \times I \to D^2$ such that $p(x \times 0) = 0$ (the origin of D^2) and $p(x \times 1) = x$.

(b) Let X be a space and let $f : S^1 \to X$ be a continuous map.
Show that f is homotopic to a constant map c if and only if f can be extended to a continuous map $\hat{f} : D^2 \to X$ (i.e. $\hat{f} \cdot i = f$ where $i : S^1 \to D^2$ is the inclusion map).

PART II.

Do three of the following:

1. (a) Let $f, g : X \to Y$ be homotopic maps. Let $x_0 \in X$.
Considering the trace of x_0 under the homotopy between f and g describe how the induced homomorphisms $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ and $g_* : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$ are related.

(b) Let A be a deformation retraction of X. Let $x_0 \in A$. Prove that the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by inclusion $i : A \to X$ is an isomorphism.

(Note: You can *not* assume that A is a *strong* deformation retraction of X).

2. (a) Let X be R^3 with the x-axis and y-axis removed. Compute $\pi_1(X)$.

(b) Let Y be the subspace of R^3 consisting of S^2 together with the z-axis. Compute $\pi_1(Y)$.

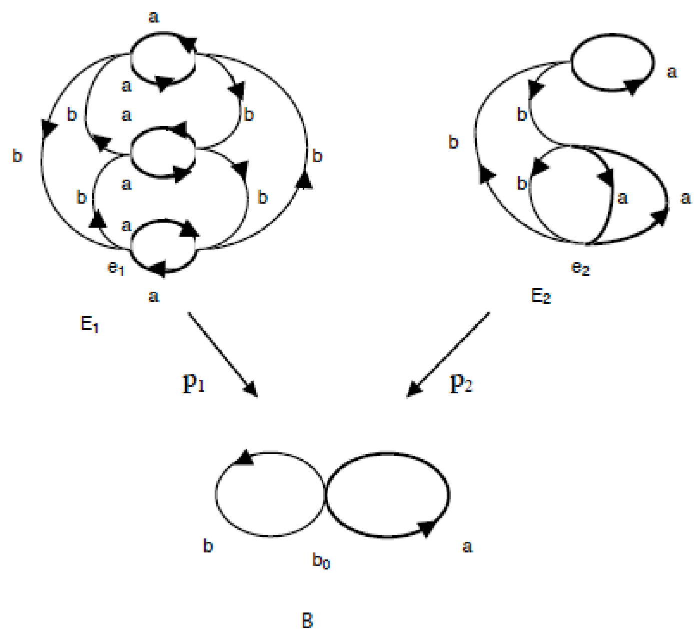
(c) The Klein Bottle K can be obtained from two copies of the Mobius band by gluing their boundary circles together via a homeomorphism. Use this description of K to find a presentation for $\pi_1(K)$.

3. (a) List all (connected) covering spaces of S^1 and list all (connected) covering spaces of P^2.

(b) Construct all *regular* connected covering spaces of $S^1 \vee P^2$ (the wedge of S^1 and P^2). Carefully explain why your list is complete. Describe (by pictures) the action of the group of Deck-transformations on the covering spaces.

(c) Construct a connected non-regular covering space of $S^1 \vee P^2$.

4. Consider the following covering spaces $p : E_i \to B$ of the wedge B of two circles, where the 1-cells labeled a, b in E_i (subscripts are omitted) are the lifts of the 1-cells labeled a and b in B.



Let $A = [a]$ and $B = [b]$ be generators of $\pi(B, b_0)$.

(a) Give a presentation of the group $p_{1*}\pi(E_1, e_1)$ as a subgroup of $\pi(B, b_0)$.

Describe the action of the group G_1 of deck transformations on E_1 and give a presentation (in terms of generators and relations) for G_1.

(b) Give a presentation of the group $p_{2*}\pi(E_2, e_2)$ as a subgroup of $\pi(B, b_0)$.

Describe the action of the group G_2 of deck transformations on E_2 and give a presentation (in terms of generators and relations) for G_2.