TOPOLOGY QUALIFYING EXAM   August 2010

Notations: X, Y, ... are topological spaces, R^n Euclidean n-space, S^n unit n-sphere in R^{n+1}.

PART I.

Do two of the following:

1.(a) Show that a space X is not connected if and only if there exists a continous surjection $\phi : X \to \{0, 1\}$ to the two-point discrete space {0,1} .

(b) Use (a) to show: If $f : X \to Y$ is a continuous map and X is connected then the image $f(X)$ is connected.

2. Suppose that f: X $\to$ Y is a homotopy equivalence.

(a) If X is path connected is Y path connected? (Give a proof or counterexample).

(b) If X is compact is Y compact? (Give a proof or counterexample).

3. Suppose $X = U \cup V$, where U, V are non-empty open sets. Let $f : [0, 1] \to X$ be a continuous map. Show: There are $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = 1$ such that for each $0 \leq i \leq n - 1$, either $f([t_i, t_{i+1}]) \subset U$ or $f([t_i, t_{i+1}]) \subset V$.

PART II.

Do three of the following (indicate which ones you are doing).

1. Determine whether there are retractions $r : X \to A$ in the following cases. In each case where there is a retraction, is there a deformation retraction?
Justify your answers.

(a) $X = \mathbb{R}^3 - (0,0,0)$ and A is the cylinder $x^2 + y^2 = 1$.

(b) $X = D^2 \vee S^1$ (where $\vee$ is the wedge) and $A = D^2$.

(c) X is a disk with two holes and A is one of its boundary circles.

(d) X is a Moebius band and A is its boundary circle.

2. Let X be the space that is obtained from the torus $T = S^1 \times S^1$ by attaching a disk D^2 under a homeomorphism $h : \partial D^2 \to S^1 \times x_0$, where x_0 is a point in S^1. Does the identity map $\iota : T \to T$ extend to a continuous map $X \to T$? Either prove that no extension exists or construct an extension.

3. Let G be the set of Deck transformations (covering transformations) of a covering $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$.

(a) Show that G is a group under composition.

(b) Let $H = p_* \pi(\tilde{X}, \tilde{x}_0) \subset \pi(X, x_0)$. Construct an epimorphism $\phi : N(H) \to G$ with kernel H.

(Here $N(H)$ is the normalizer of H in $\pi(X, x_0)$. Define ϕ, show it is well-defined, show ϕ is surjective, show $ker(\phi) = H$. You need not show that ϕ is a homomorphism).

4. Let $\mathbf{Z} * \mathbf{Z_2} = \langle a\,,\, b \,|\, b^2 \rangle$ be a presentation of the fundamental group of $X = S^1 \vee P^2$

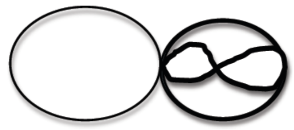

Figure 1: $X = S^1 \vee P^2$

For the subgroup H below construct the covering space $\tilde{X}$ by sketching a good picture for $\tilde{X}$ and explaining (in your picture) how it covers X.

In each case give a group presentation for the group G of covering transformations of the covering $p : \tilde{X} \to X$ and describe (using your picture) the action of G on $\tilde{X}$.

(a) H is the smallest normal subgroup containing b.

(b) H is the smallest normal subgroup containing a.

(c) H is the smallest normal subgroup containing a^2 and b.