TOPOLOGY QUALIFYING EXAM   January 2011

PART I.

Do two of the following:

1. Prove: A compact Hausdorff space is regular.

2. (a) Show: A topological space Y is Hausdorff if and only if the subset $\triangle = \{(y, y) \in Y \times Y \mid y \in Y\}$ is closed in $Y \times Y$.

(b) Let $f, g : X \to Y$ be a continuous map of a space X to a Hausdorff space Y. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.
(Hint: Consider a certain map $X \to Y \times Y$ and use (a)).

3. Let A be a retract of a topological space X.

(a) Show: If X is connected then A is connected.

(b) Show: If X is compact then A is compact.

(c) Show: If X is Hausdorff then A is a closed subspace of X.

PART II.

Do three of the following (indicate which ones you are doing).

1. (a) Let $f, g : X \to Y$ be homotopic maps. Describe how the induced homomorphisms $f_\star : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ and $g_\star : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$ are related.

(b) Let A be a deformation retraction of X. Let $x_0 \in A$. Prove that the homomorphism $i_\star : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by inclusion $i : A \to X$ is an isomorphism.

(Note: You may *not* assume that A is a *strong* deformation retract of X).

2. Let S be the surface obtained from the closed orientable surface of genus 2 by removing the interior of a small disk and let C be the boundary circle of S.

(a) Find a presentation of $\pi(S, s_0)$, where the base point s_0 is on C.

(b) Let γ be a generator of $\pi(C, s_0)$. Write γ as a word in terms of the generators of $\pi(S, s_0)$.

(c) Prove that there is no retraction $r : S \to C$.

3. Let $\tilde{X}$ and $\tilde{Y}$ be simply connected covering spaces of path-connected and locally path-connected spaces X and Y.

Show: If X and Y are homotopy equivalent then $\tilde{X}$ and $\tilde{Y}$ are homotopy equivalent.

(You may use the following fact: If $f : A \to B$, $g : B \to A$ are such that gf is homotopic to a homeomorphism and fg is homotopic to a homeomorphism, then A and B are homotopy equivalent).

4. Let $\mathbf{Z} * \mathbf{Z_2} = \langle a\,,\, b \,|\, b^2 \rangle$ be a presentation of the fundamental group of $X = S^1 \vee P^2$

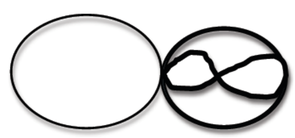

Figure 1: $X = S^1 \vee P^2$

(a) Construct a *regular* 3-sheeted covering space $\tilde{X}$ of X. Sketch a schematic picture and indicate on the picture how $\tilde{X}$ covers X.

Fix base points for X and $\tilde{X}$ and let H_1 be the subgroup corresponding to the covering. Find generators of H_1 and express them in terms of a and b.

(b) Construct a *non-regular* 3-sheeted covering space $\tilde{Y}$ of X. Sketch a schematic picture and indicate on the picture how $\tilde{Y}$ covers X.

Fix base points for X and $\tilde{Y}$ and let H_2 be the subgroup corresponding to the covering. Find generators of H_2 and express them in terms of a and b.