Topology Qualifying Exam **Fall 2019**

Make sure you explicitly mention any theorems you use and explain why the relevant hypotheses are satisfied. Unless otherwise stated, all functions are assumed to be continuous.

Part I

Attempt two of the three problems in this section (mark which ones you are attempting)

1. Let X and Y be locally compact and Hausdorff topological spaces. Recall that the *one-point compactification* of X is the topological space $\hat{X} = X \cup \{\infty\}$ topologized so that open sets consist of open sets $U \subset X$ and sets of the form $(X \backslash C) \cup \{\infty\}$, where $C \subset X$ is compact. Next, recall that a function $f : A \to B$ is *proper* if for each compact $C \subset B$, $f^{-1}(C) \subset A$ is compact.

   Let $\hat{X}$ be the one-point compactification of X, let $\hat{Y}$ be the one-point compactification of Y, and let $f : X \to Y$. Prove that f can be extended to a continuous map $\hat{f} : \hat{X} \to \hat{Y}$ if and only if f is proper.

2. Let X, Y, and Z be topological spaces and let $f : X \to Z$ and $g : Y \to Z$ be continuous functions. The fiber product is defined as

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

   topologized using the subspace topology of the product topology on $X \times Y$.

   (a) Prove that the projection maps $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ are continuous.

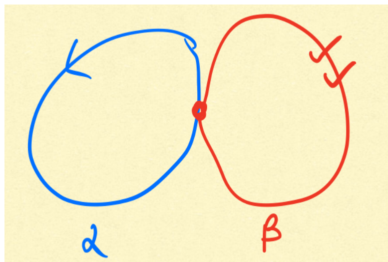
   (b) Prove that if W is a topological space and $h : W \to X \times_Z Y$ is a function then h is continuous if and only if both $p_X \circ h$ and $p_Y \circ h$ are continuous.

3. Let (X, d) be a metric space and let A be a compact subspace. Prove that A is both closed and bounded. For this problem you may assume without proof that for any $y \in X$ the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, y)$ is continuous.

Part II

Attempt three of the four problems in this section (mark which ones you are attempting)

1.



Let $G = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$ and let $X = S^1 \vee S^1$. Let x_0 be the wedge point of X and recall that $\pi_1(S^1 \vee S^1, x_0) \cong G$ via the map taking the homotopy classes of α and β in the figure above to a and b

(a) Construct a connected 3-fold irregular cover $p : \tilde{X} \to X$ of X. Make sure to explain why your cover is irregular.

(b) Pick a point $\tilde{x}_0 \in \tilde{X}$ that covers x_0 and write a presentation for $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ using the generators a and b. What is the index of this subgroup?

(c) Recall that a subgroup H of G is *self-normalizing* if the normalizer $N_G(H)$, of H in G is equal to H. Explain why $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a self-normalizing subgroup of G.

2. $\mathbb{RP}^2$ can be constructed as follows: let $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the disk and let $S^1 = \partial D^2 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle. $\mathbb{RP}^2$ can then be formed by attaching D^2 to S^1 via the 2:1 map $f : \partial D^2 \to S^1$ given by $f(z) = z^2$

   Use the above description and Van Kampen's theorem to find a presentation for $\pi_1(\mathbb{RP}^2)$. Make sure to be very clear which sets you are decomposing $\mathbb{RP}^2$ into and explain why they satisfy all the relevant hypotheses.

   This is a very common group, which group is it?

3. Let X be a path connected space, $p : \tilde{X} \to X$ be a covering, and $A \subset X$ a path connected subspace.

   (a) Let $x_0 \in A$ and let $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$. If $\tilde{A}$ is the component of $p^{-1}(A)$ that contains $\tilde{x}_0$ prove that $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering.

   (b) Let $\iota : A \to X$ be the inclusion map. Prove that the set of connected components of $p^{-1}(A)$ is in bijection with the coset space $\pi_1(X, x_0)/\iota_*(\pi_1(A, x_0))$.

4. Let X be a topological space and let $A \subset X$ be a subspace. Suppose that $f_t : X \to X$ is a deformation retract from X to A. Prove that the inclusion $\iota : A \to X$ is a homotopy equivalence.