Topology Qualifying Exam **Spring 2020**

Make sure you explicitly mention any theorems you use and explain why the relevant hypotheses are satisfied. Unless otherwise stated, all functions are assumed to be continuous.

Part I

Attempt both problems in this section.

1. Let X be a topological space and let $\Delta : X \to X \times X$ given by $\Delta(x) = (x, x)$ be the diagonal embedding.

   a. Prove that if $X \times X$ is equipped with the product topology then Δ is continuous.

   b. Prove that X is Hausdorff if and only if the image $\Delta(X) \subset X \times X$ is closed.

2. Let X and Y be first countable topological spaces

   a. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in Y such that $a_n \to a$. Prove that

   $$\{a_n \mid n \in \mathbb{N}\} \cup \{a\}$$

   is a compact subset of Y.

   b. Let $A \subset Y$ and let x be a limit point of A. Prove that there is a sequence $\{a_n\}_{n=1}^{\infty}$ in A so that $a_n \to a$. Hint: This is false if Y is not first countable.

   c. Assume that Y is Hausdorff and let $f : X \to Y$ be a proper function (i.e. $K \subset Y$ compact implies $f^{-1}(K) \subset X$ is compact). Prove that if $C \subset X$ is closed then $f(C) \subset Y$ is closed. Hint: For first countable spaces compactness implies sequential compactness.

Part II

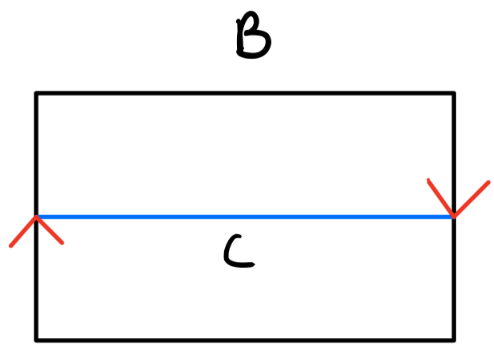
Attempt three of the four problems in this section (mark which ones you are attempting)

1. Let X be a path connected and locally path connected space and let $p : \tilde{X} \to X$ be a universal covering.

   a. Let $A \subset X$ be a path connected subset, Prove that if $\tilde{A}$ is a path component of $p^{-1}(A)$ then $p' := p|_{\tilde{A}} : \tilde{A} \to A$ is a covering of A.

   b. Let $q \in A$ and $\tilde{q} \in \tilde{A}$ so that $p'(\tilde{q}) = q$. Prove that $p'_*(\pi_1(\tilde{A}, \tilde{q})) = \ker(\iota_*)$, where $\iota : A \to X$ is the inclusion map.

2. Let K be real projective 2-space and recall that K can be constructed by attaching a disk D to the boundary curve of a Möbius band B (pictured below)



   a. Compute $\pi_1(K)$

   b. Let C be the center curve of the Möbius band and let K' be the space obtained by taking two copies of K and identifying the curve C in both copies. Compute $\pi_1(K')$.

   c. Describe the universal cover of K'.

3. Let X be a topological space and let G be a group acting on X via homeomorphisms. Recall that such a group action is a *covering space action* if for each $u \in X$ there is an open neighborhood $U \ni u$ such that if $g_1, g_2 \in G$ and $g_1(U) \cap g_2(U) \neq \emptyset$ then $g_1 = g_2$.

   a. Let $X = \mathbb{R}^2 \backslash \{(0,0)\}$ and let $G = \mathbb{Z}$ and let G act on X so that the generator $1 \in \mathbb{Z}$ acts via $1 \cdot (x, y) = (2x, 2y)$. Prove that this is a covering space action.

   b. Prove that if we let $X = \mathbb{R}^2$ and we extend the action of G using the same formula as in part a. then the action is no longer a covering space action.

   c. Let Y the space of orbits of the action from part a. and let $p : X \to Y$ be the map that takes a point in X to its orbit. What is $\pi_1(Y)$?

4. Recall that a topological space X is *contractible* if the identity map $1_X : X \to X$ is homotopic to a constant map.

   a. Show that every contractible space is path connected.

   b. Suppose that if X is contractible and Y is path connected. Prove that if $f_1, f_2 : X \to Y$ are maps then f_1 and f_2 are homotopic maps.

   c. Find a counterexample to b. if we drop the hypothesis that Y is path connected.