# Topology Qualifying Exam **Fall 2021**

Make sure you explicitly mention any theorems you use and explain why the relevant hypotheses are satisfied. Unless otherwise stated, all functions are assumed to be continuous.

## Part I

*Attempt both problems in this section.*

1. Let $X = [-1, 1] \subseteq \mathbb{R}$. For $a < 0 < b$ consider the subsets

$$L_b = [-1, b), \quad R_a = (a, 1], \quad U_{a,b} = (a, b).$$

Consider the topology on $X$ generated by the basis $\mathcal{B} = \{L_b, R_a, U_{a,b} \mid a < 0 < b\}$. Answer the following questions, providing proofs or counterexamples as appropriate.

a. Is the following function $f : X \to X$ continuous?

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

b. Is the following sequence convergent?

$$t_n = \begin{cases} 0 & n \text{ even} \\ \frac{1}{2} & n \text{ odd} \end{cases}$$

c. Show that $X$ compact but not Hausdorff.

2. Give $\mathbb{N}$ the discrete topology and let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. Let $Y$ be a topological space and let $f : \mathbb{N} \to Y$. This function naturally gives rise to a sequence $\{f(n)\}_{n \in \mathbb{N}}$ in $Y$.

Prove that the above sequence converges if and only if $f$ can be continuously extended to $\hat{f} : X \to Y$.

## Part II

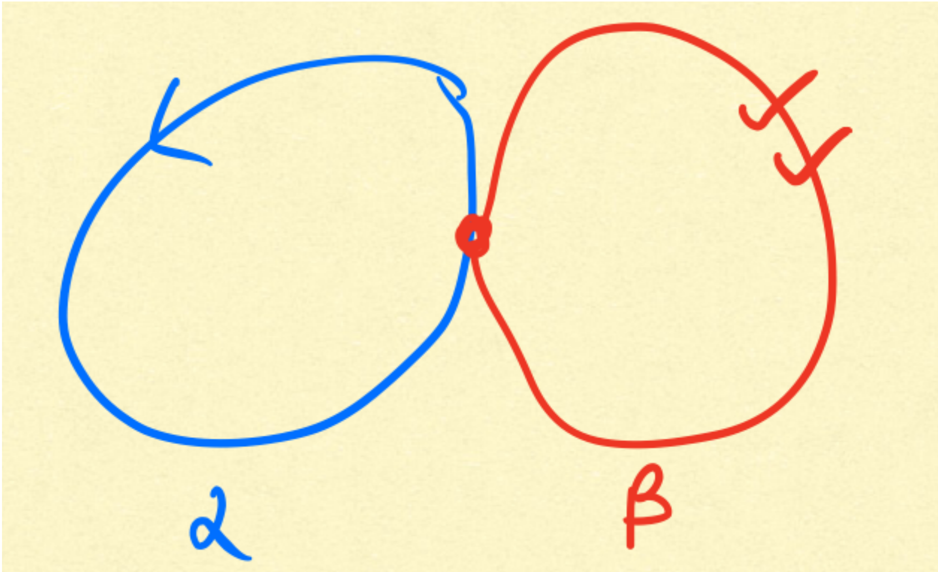
*Attempt three of the four problems in this section (mark which ones you are attempting)*

1. Let $X$ be a path connected and locally path connected space and let $p : \tilde{X} \to X$ be a universal covering.

a. Let $A \subset X$ be a path connected subset, Prove that if $\tilde{A}$ is a path component of $p^{-1}(A)$ then $p' := p|_{\tilde{A}} : \tilde{A} \to A$ is a covering of $A$.

b. Let $q \in A$ and $\tilde{q} \in \tilde{A}$ so that $p'(\tilde{q}) = q$. Prove that $p'_*(\pi_1(\tilde{A}, \tilde{q})) = \ker(\iota_*)$, where $\iota : A \to X$ is the inclusion map.

2. Let $X = S^1 * S^1$ be a wedge of two circles with wedge point $q$. $\pi_1(X, 1)$ is generated by the homotopy classes of the curves $\alpha$ and $\beta$ (see figure below).




a. Find a 4 sheeted **irregular**[1] cover $\tilde{X}$ of $X$

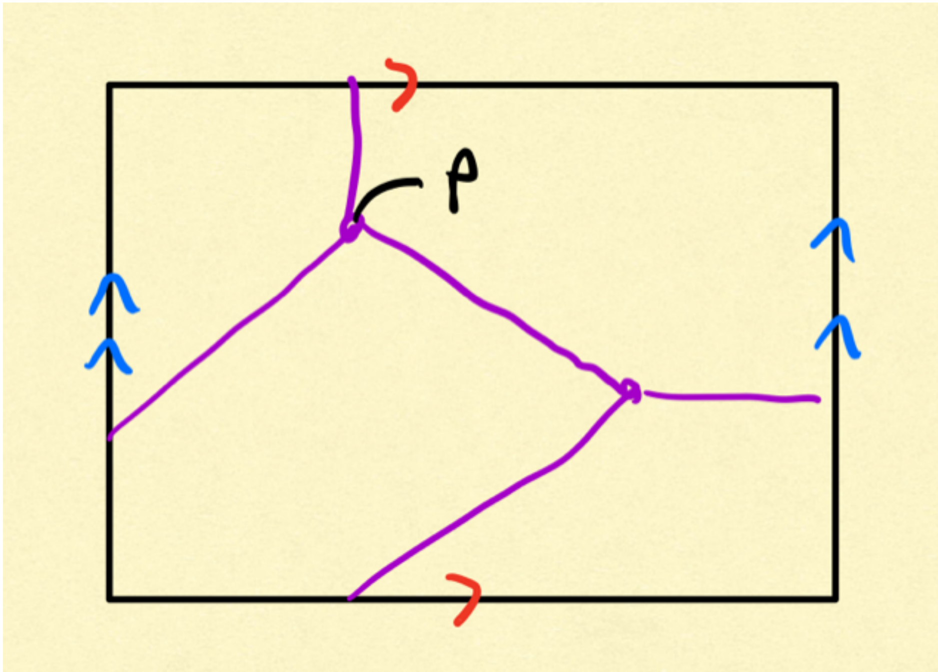
b. Let $p : \tilde{X} \to X$ be the covering map to the cover you constructed in a. Let $q$ be the wedge point of $X$ and pick a lift $\tilde{q}$ of $q$ in your cover $\tilde{X}$. Write down a generating set (with respect to $[\alpha]$ and $[\beta]$) for the subgroup $p_*(\pi_1(\tilde{X}, \tilde{q})) \subset \pi_1(X, q)$.

c. Find a 4 sheeted **regular** cover $\tilde{Y}$ of $X$.

d. Let $f : \tilde{Y} \to X$ be the covering map to the cover you constructed in c. Let $q$ be the wedge point of $X$ and pick a lift $\tilde{q}$ of $q$ in your cover $\tilde{Y}$. What is the quotient group $\pi_1(X, q)/f_*(\pi_1(\tilde{Y}, \tilde{q}))$

[1] Note that regular (resp. irregular) covers are sometimes called normal (resp. non-normal)

3. The figure below shows a graphs $G$ (purple) embedded in a torus $T$.




   a. Draw a collection of curves based at $p$ whose homotopy classes generate $\pi_1(G,p)$
   b. Let $\iota : G \to T$ be the inclusion map. Prove that $\iota_* : \pi_1(G,p) \to \pi_1(T,p)$ is a surjection.
   c. Write down a set of curves whose homotopy classes normally generate the subgroup $\ker \iota_* \subset \pi_1(G,p)$.

4. The van-Kampen theorem says that if $X$ is a topological space and $U$ and $V$ are open subsets of $X$ with path connected intersection such that $U \cup V = X$ then for any $p \in U \cap V$ $\pi_1(X,p) \cong (\pi_1(U,p) * \pi_1(V,p))/N$, for some subgroup $N$.
   a. Prove that the hypotheses of the theorem cannot be weakened to allow $U \cap V$ to be non path connected
   b. Prove that the hypotheses of the theorem cannot be weakened to allow $U$ or $V$ to be non open.