Topology Qualifying Exam **Spring 2022**

Make sure you explicitly mention any theorems you use and explain why the relevant hypotheses are satisfied. Unless otherwise stated, all functions are assumed to be continuous. Attempt all five problems

Part I

1. Let $a, b \in \mathbb{Z}$, $a \neq 0$, and let $U_{a,b} = \{an + b \mid n \in \mathbb{Z}\}$ (i.e. $U_{a,b}$ is an arithmetic progression). The topology generated by the $U_{a,b}$ is called the *profinite topology*

   a. Prove that $\{0\}$ is a closed set in this topology.

   b. Define $f : \mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ by $f(n) = [2n]$, where $[m]$ indicates reduction of m mod 8. Prove that if $\mathbb{Z}/8\mathbb{Z}$ is given the discrete topology and $\mathbb{Z}$ is given the profinite topology then f is continuous.

   c. Let $g : \mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ be given by

   $$g(n) = \begin{cases} [n], n \neq 0 \\ 1, n = 0 \end{cases}$$

   Prove that if $\mathbb{Z}/8\mathbb{Z}$ is given the discrete topology and $\mathbb{Z}$ is given the profinite topology then g is not continuous.

2. Every number $x \in [0,1]$ can be written *triadically* i.e. $x = \sum_{i=1}^{\infty} c_i 3^{-i}$, where $c_i \in \{0,1,2\}$.

   Here is a fact you may use for this problem: if $x_j = \sum_{i=1}^{\infty} c_i^j 3^{-i}$, $x = \sum_{i=1}^{\infty} c_i 3^{-i}$ then $x_j \to x$ if and only if for each $N \in \mathbb{N}$ there is $M \in \mathbb{N}$ so that if $j \geq M$ then $c_l^j = c_l$ for $l \leq N$.

   a. The *middle third cantor* set can be defined as

   $$C = \left\{ \sum_{i=1}^{\infty} c_i 3^{-i} \mid c_i \neq 1 \ \forall i \right\}$$

   Give $C \subset [0,1]$ the subspace topology. Prove that C is closed

   b. An *isolated point* of a topological space X is a point $p \in X$ so that $\{p\}$ is open in X. Prove that C contains no isolated points.

Part II

1. This question is about the *monodromy action* More specifically, let X be a topological space, and let $p \in X$, let $\tilde{X} \xrightarrow{f} X$ be a covering and let $\tilde{p} \in \tilde{X}$ so that $f(\tilde{p}) = p$. Let $F = f^{-1}(p)$. For $[\gamma] \in \pi_1(X,p)$ and $q \in F$ define $q \cdot [\gamma]$ to be the endpoint of the lift of γ to $\tilde{X}$ starting at q.

   a. Prove that the above formula defines a right action of $\pi_1(X,p)$ on F. I.e. prove that

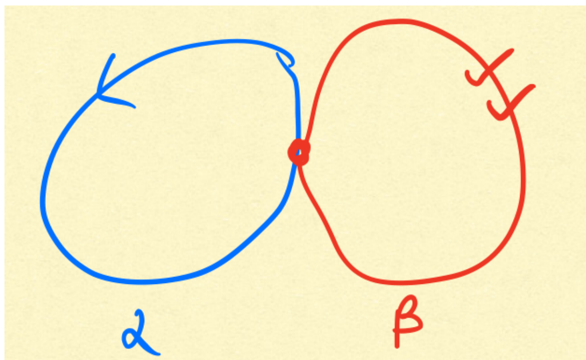
      i. $q \cdot [c] = q$ for each $q \in F$ when $[c]$ is the constant loop and

ii. if $[\gamma], [\gamma'] \in \pi_1(X,p)$ then $(q \cdot [\gamma]) \cdot [\gamma'] = q \cdot ([\gamma][\gamma'])$

b. Let $K \leq \pi_1(X,p)$ be the subgroup of elements so that $[\gamma] \in K \Leftrightarrow \tilde{p} \cdot [\gamma] = \tilde{p}$. Prove that $K = f_*(\pi_1(\tilde{X}, \tilde{p}))$

c. Let $K' \leq \pi_1(X,p)$ be the subgroup of elements so that $[\gamma] \in K \Leftrightarrow q \cdot [\gamma] = q$ $\forall q \in F$. Prove that $K' \leq f_*(\pi_1(\tilde{X}, \tilde{p}))$ with equality if and only if $\tilde{X} \xrightarrow{f} X$ is a regular cover.

2. Let X be a wedge of two circles shown below. This problem asks you to construct some covering space of X for each covering, make sure to label the lifts of the wedge point and the curves α and β.



a. construct a 4-fold cover of X whose deck group is $\mathbb{Z}/4\mathbb{Z}$

b. Construct a 4-fold cover of X whose deck group is $\mathbb{Z}/2\mathbb{Z}$

c. Construct a 6-fold cover of X whose deck group is S_3. Hint S_3 is generated by the permutations (12) and (123).

3. Recall the definition of the wedge product of topological spaces. Let X and Y be connected locally contractible (i.e. every point has a contractible neighborhood) topological spaces and let $p \in X$ and $q \in Y$. Consider the equivalence relation $\sim$ on $X \sqcup Y$ generated by $a \sim b$ iff $a = p$ and $b = q$. Let $X * Y$ be space of equivalence classes of this relation topologized with the quotient topology. Let $* \in X * Y$ denoted the equivalence class $[p] = [q]$. Prove that $\pi_1(X * Y, *) \cong \pi_1(X,p) * \pi_1(Y,q)$.

If you use Van-Kampen's theorem make sure to be very clear about the sets you use in your decomposition and explain why the satisfy the hypotheses of the theorem.