Topology Qualifying Exam **Fall 2023**

Make sure you explicitly mention any theorems you use and explain why the relevant hypotheses are satisfied. Unless otherwise stated, all functions are assumed to be continuous. Attempt all five problems

Part I

1. Let $\mathbb{R}[x]$ be the set of polynomials with real coefficients in 1 variable. Each $f \in \mathbb{R}[x]$ can be thought of as a function $f : \mathbb{R} \to \mathbb{R}$. Define $U_f = \{x \in \mathbb{R} \mid f(x) \neq 0\}$ and let $\mathcal{B} = \{U_f \mid f \in \mathbb{R}[x]\}$

   a. Prove that $\mathcal{B}$ is the basis for a topology on $\mathbb{R}$. This topology is called the *Zariski topology.*

   b. Prove that the Zariski topology is T_1 (i.e. singletons in X are closed) but not Hausdorff.

   c. Prove that for each $f \in \mathbb{R}[x]$ then function $f : \mathbb{R} \to \mathbb{R}$ is continuous if we put the Zariski topology on $\mathbb{R}$.

   d. Let τ be a topology on $\mathbb{R}$. Suppose that for each $f \in \mathbb{R}[x]$ that $f : \mathbb{R} \to \mathbb{R}$ is continuous when we use τ as the topology on the domain and the Zariski topology on the codomain. Prove that τ is finer than the Zariski topology.

2. Let X be a Hausdorff topological space and let $A \subset X$ be compact.

   a. Prove that for each $b \notin A$ there are open subsets U and V of X so that $A \subset U$, $b \in V$, and $U \cap V = \emptyset$.

   b. Prove that if X is compact and Hausdorff then X is regular.

Part II

1. Recall that a topological space X is *semi-locally simply connected* if for each $p \in X$, there is a neighborhood $p \in U \subset X$ so that the map $\iota_* : \pi_1(U, p) \to \pi_1(X, p)$ induced by the inclusion $\iota : U \hookrightarrow X$ is trivial.

   Let X be a connected topological space and $\tilde{X}$ a simply connected covering of X with covering map $p : \tilde{X} \to X$. Prove that X is semi-locally simply connected.

2. Let $F_2 = \langle a, b \rangle$ be the free group on 2 generators. For this problem, you will probably want to use the fact that F_2 is the fundamental group of the wedge of 2 circles.

   a. Provide generators for a subgroup H of F_2 that is normal and has index 3 in F_2. Make sure to explain why your example is normal and has the correct index.

   b. Provide generators for a subgroup G of F_2 that has index 4 in F_2 and whose normalizer in F_2 has index 2 in F_2. Again, make sure to explain why your answer is correct.

3. Let X be a topological space and let U and V be open subsets of X that contain a common point x_0. Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$, and $j_2 : V \to X$ be the corresponding inclusion maps. Suppose these sets have the following properties

   - $X = U \cup V$
   - U, V, and $U \cap V$ are connected

   a. Suppose that $(i_1)_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is an isomorphism. Prove that there is a homomorphism $\Phi : \pi_1(X, x_0) \to \pi_1(V, x_0)$ such that $\Phi \circ (j_2)_*$ is the identity map

   b. Suppose that $(i_1)_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto. Prove that $(j_2)_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ is also onto.