Topology Qualifying Exam August 2008

There are two parts to this exam. The exam continues on the next page.

Part 1. **Do any three of the following four problems.**

1. Prove that the graph of a function $f : [0,1] \to \mathbb{R}$ is compact if and only if f is continuous. Give an example of a discontinuous function $g : [0,1] \to \mathbb{R}$ with a graph that is closed but not compact.

2. Give an example of a subset of $\mathbb{R}^2$ that is connected but not path-connected.

3. Let X be a topological space. A subset $A \subset X$ is a *retract* if there exists a continuous map

$$r : X \to A$$

so that the restriction of r to A is the identity map. Prove that if X is Hausdorff, then A must be a closed subset.

4. Let (X, d) be a non-empty complete metric space. A map $f : X \to X$ is a *contraction* if there is a real number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$. Show that if f is a contraction, then there is a unique point $x \in X$ such that $f(x) = x$.

Part 2. **Do any three of the following four problems. All spaces are assumed to be path connected, locally path connected, and semi-locally simply connected.**

1. Let K be the Klein bottle, M the Klein bottle with an open disk removed and C the boundary curve of M.

   (a) Find the fundamental groups of K, M and C.

   (b) Is C a deformation retract of M?

2. Let X be the wedge product $X = Y_1 \vee Y_2$ of two circles Y_1 and Y_2 (i.e, X is obtained by taking the quotient of $Y_1 \cup Y_2$ with two points, $y_1 \in Y_1$ and $y_2 \in Y_2$, identified). Let x_0 be the image of y_1 and y_2 in X under the quotient map. Let a and b be elements of the fundamental group $\pi_1(X, x_0)$ represented by closed paths going once around Y_1 and Y_2, respectively.

   (a) Describe the covering (using pictures) associated to the subgroup N of $\pi_1(X, x_0)$ generated by a^2, ab, and b^2.

   (b) Show that N is a normal subgroup of $\pi_1(X, x_0)$.

   (c) What is the group of deck transformations?

   Hint: Some may find it easier to do (b) before (a).

3. Let $X = S^1 \vee S^2$ be the wedge product of the circle S^1 and the sphere S^2 (the definition of wedge product is explained in Part 2: Problem 2). Find the universal covering of X and describe its deck transformations.

4. Let $\rho : (\widetilde{X}, \widetilde{x_0}, \rightarrow (X, x_0)$ be a covering $\widetilde{X} \rightarrow X$ with $\rho(\widetilde{x_0}) = x_0$. Let ϕ : $(X, x_0) \rightarrow (X, x_0)$ be a homeomorphism of X to itself with $\phi(x_0) = x_0$. Prove that ϕ lifts to a homeomorphism

$$\widetilde{\phi} : (\widetilde{X}, \widetilde{x_0}) \rightarrow (\widetilde{X}, \widetilde{x_0})$$

   (i.e., $\rho \circ \widetilde{\phi} = \rho$ and $\widetilde{\phi}(\widetilde{x_0}) = \widetilde{x_0})$ if and only if the induced map

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

   preserves the image of

$$\rho_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \rightarrow \pi_1(X, x_0)$$

   (i.e., if K is the image of ρ_*, then $\phi_*(K) = K$).