**TOPOLOGY QUALIFYING EXAM May 2008**

**PART I.**

**Do any two of the following:**

1. (a) Prove that a topological space $X$ is Hausdorff if and only if the subset $D = \{(x, x) \in X \times X \,|\, x \in X\}$ is closed in $X \times X$ (in the product topology).

(b) Let $X$ be a Hausdorff space and let $f : X \to X$ be a continuous function. Prove that $F = \{x \in X \,|\, f(x) = x\}$ is closed in $X$ .

2. (a) Give an example to show that the projection map $p : X \times Y \to Y$ need not be a closed map. (Hint: Try $X = Y = \mathbf{R}$).

(b)Let $X$ be compact and $Y$ be Hausdorff. Show that the projection $p : X \times Y \to Y$ is a closed map.

(Hint: If $A$ is closed in $X \times Y$ and $y \in Y - p(A)$ , start by constructing a cover of $p^{-1}(y)$ by basic open sets in $X \times Y - A$).

3. Let $p : X \to Y$ be a quotient map and $h : X \to Z$ a continous map such that for each $y \in Y$ , $h$ is constant on $p^{-1}(y)$ . Show that there is a continuous map $k : Y \to Z$ such that $h = k \cdot p$.

**PART II.**

**Do any four of the following:**

**1**. Determine which of the four spaces (i), (ii), (iii) (iv) are homotopy equivalent. (Give detailed arguments).

(i) $A$ = torus with a small disk removed
(ii) $B$ = 2-sphere together with an arc connecting the North pole to the South pole.
(iii) $C$ = torus
(iv) $D$ = Moebius band with a small disk removed.

**2**. Denote by $m_i$ a Moebiusband and by $c_i$ its boundary curve ($i = 1, 2$). Let $X$ be the quotient space obtained from $m_1$ and $m_2$ by identifying $c_1$ and $c_2$ by a homeomorphism and let $x_0$ be a point on $c = c_1 = c_2 \subset X$.

(a) Obtain a presentation of $\pi_1(X, x_0)$.

(b) Does there exist a retraction $r : X \to c$ ? (Either describe such a retraction or give a detailed proof that no such retraction exist).

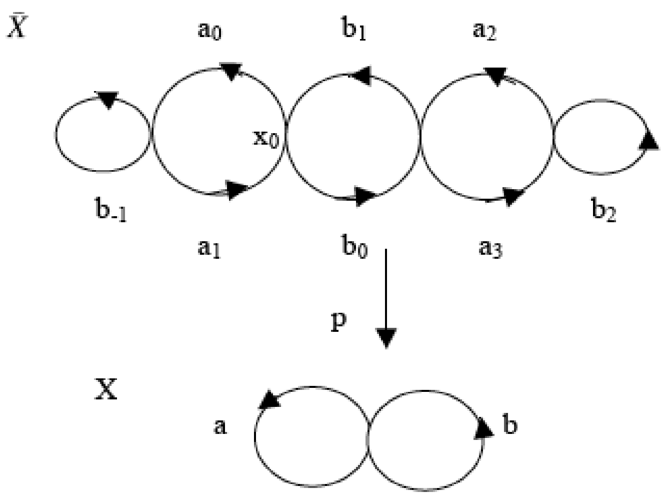
**3**. Let $X = S^1 \cup_f D^2$ be the adjunction space , where $f : Bd(D^2) = S^1 \to S^1$ is the map $f(z) = z^3$ .

(a) Compute $\pi_1(X)$ .

(b) Describe the universal covering space $\tilde{X}$ of $X$.

(c) What is the group $G$ of Decktransformations and how does $G$ act on $\tilde{X}$ ?

**4**. Consider the following covering space $p : \tilde{X} \to X$ of the wedge $X$ of two circles, where the 1-cells labeled $a_i$, $b_j$ are the lifts of the 1-cells labeled $a$ and $b$, resp.




Let $A = [a]$ and $B = [b]$ be generators of $\pi(X, v)$ , where $v$ is the wedge point.

(a) Give a presentation of the group $p_*\pi(\tilde{X}, x_0)$ as a subgroup of $\pi(X, v)$ .

(b) Give a presentation (in terms of generators and relations) for the group $G$ of Deck transformations of $\tilde{X}$.

**5**. Represent $\mathbf{Z} * \mathbf{Z} = \langle a\,,\, b \rangle$ as the fundamental group of $X = S^1 \vee S^1$ with wedge point $x_0$.

Let $\tilde{X}$ be the covering space of $X$ corresponding to the smallest normal subgroup $H$ of $\pi(X, x_0)$ that contains the elements $aba^{-1}b^{-1}$ , $a^2$ and $b^2$ .

(a) Give a presentation of the Deck transformation group $G$.

What is the cardinality of the fiber over $x_0$?

(b) Draw a picture of $\tilde{X}$, labeling clearly the (directed) edges that cover the edges of $X$ corresponding to $a$ and $b$.