Lecture 4

Suchandan Pal

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Recall: functor of points.

If $K$ is a field and $a \subset k[x_1,\ldots,x_n]$ is an ideal then if $k \subset L$, $L$ a field. Given $f_1,\ldots,f_n \in k[x_1,\ldots,x_n]$ we can ask for the zero in $L^n$. the solutions to $a$ over $L$ are in bijection with the $k$-algebra homomorphisms $k[x_1,\ldots,x_n] \rightarrow L$, $x_i \rightarrow a_i$.

$k$-algebra homomorphisms are morphisms of schemes.

$$\text{Spec } k[x_1,\ldots,x_n]/a \leftarrow \text{Spec } L$$

In particular, take $L = \overline{k}$ and letting $X = Z(a) \subseteq A^n_k$ we get $X = \text{hom}_{\text{sch}(k)}(\overline{k}, \text{Spec } A(X))$. So we can recover $X$ from it’s “associated scheme” Spec $A(x)$. This is a better way of thinking of a variety as a scheme. Also, Spec $A(x)$ carries more information than $X$. We can also recover solutions over any field extension of $k$. This might explain the terminology “scheme”. It motivates the following definition:

If $X$ is a scheme over some field $k$, and $L$ is an extension field, then define:

$$X(L) := \text{hom}_{\text{sch}(k)}(L, X)$$

called the set of $L$ valued points of $X$.

More generally, if $X,Y$ are schemes on a scheme $S$(base).

Remark: We should distinguish this (even when $L = \overline{k}$) from the points of the scheme.

Remark: If $R$ is a ring then there exists a unique ring homomorphism $\mathbb{Z} \rightarrow R$. Spec $\mathbb{Z} \leftarrow$ Spec $R$. More generally one can show that any scheme has a unique map to Spec $\mathbb{Z}$. So Spec $\mathbb{Z}$ is a kind of universal base. Hartshorne 2.2.*

If $X,Y$ are schemes over a scheme $S$ (base) then we define $X(y) := \text{hom}_{\text{sch}(S)}(Y, X)$. These are called the set of $Y$-valued points. of $X$. 

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Example: If $E$ is $y^2 = x^3 - x$ the elliptic curve defined over $\mathbb{Q}$ then $E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 | (x, y) \in E\}$. Rational solutions.

Vista: This gives a functor associated to $X$ denoted $h_X : \text{Sch}(s) \to \text{Sets}$. $x \mapsto \text{hom}_{\text{Sch}(s)}(Y, X)$. So there is a functor from $\text{Sch}(s) \to \text{Functors}$ from $\text{Sch}(s)$ to sets. $x \mapsto \text{hom}_{\text{Sch}(s)}(\cdot, x)$.

Remark: A functor of the form $h_x$ is called a representable functor.

This gives an equivalence of $\text{Sch}(s)$ with a full subcategory of the category of functors from $\text{Sch}(s)$ to sets.

Point: Many geometrical constructions, i.e. tangent space, can be carried out using functors. See “Geometry of Schemes”.

Fibered product:
Section 2.3

Recall: If $C$ is a category and $X, Y \in \text{Obj}(C)$ then the product $X \times Y$ in $C$ is an object that satisfies a certain universal property.

i.e. $C$ is sets, $X \times Y$ is the cartesian product. We want to consider products of algebraic sets/schemes and the same over a fixed base.

**Definition 0.1.** The fibered product.

If $S \in \text{Obj}(C)$ the fibered products in $C/S := \text{Category of objects over S}$. if $x, y$ is an $S$-object $X \times Y$ with maps $\phi_1 : X \times_SY \to X, \phi_2 : X \times_SY \to Y$.
Remark: If $C$ has a terminal object $S$ then the fibered product over $S$ is the product.

A (fibered) product is unique up to unique isomorphism.

What should be products of algebraic sets? 1st idea: If $X, Y$ are affine varieties, $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ then $X \times Y \subseteq \mathbb{A}^{m+n}$

We need to check that the Zariski topology is respected.

The same does not work for projective space because the number of coordinates don’t match up: $\mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{m+n}$

What does work: Serge embedding.

Suppose $X, Y$ are affine varieties over a field $k$ and $X \times_k Y$ exists, then

There is a correspondence with varieties and finitely generated domains over $k$.

Is there a universal object in the category of finitely generated domains over $k$ which has this universal property satisfied by $A(X \times_k Y)$. Answer: Yes. $A(X) \otimes A(Y)$. In general in the category of modules over a fing $R$ there is a fibered product, the tensor product denoted $M \otimes_R N$

First: Tensor products of modules.

Let $M, N$ be modules over a ring $R$. 

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**Definition 0.2.** A map $\phi : M \times N \to L$ where $L$ is an $R$-module, is said to be $R$-bilinear if:

\[
\begin{align*}
\phi(a + b, c) &= \phi(a, c) + \phi(b, c) \\
\phi(a, b + c) &= \phi(a, b) + \phi(a, c) \\
\phi(ra, b) &= \phi(a, rb)
\end{align*}
\]

We want a universal object for such maps, i.e. an object $M \square_R N$ with a map $\psi : M \times N \to M \square_R N$ which fits into the following commutative diagram:

\[
\begin{array}{c}
M \times N \xrightarrow{\phi} M \square_R N \\
\downarrow \psi \quad \exists \theta \\
L
\end{array}
\]

So $\{R$-bilinear maps $M \times N \to L \} \leftrightarrow \{R$-module homomorphisms $M \square_R N \to L \}$

Idea to construct $M \square N$:

Consider all pairs $(m, n)$ and force the conditions.

**Definition 0.3.** We will call $M \square_R N M \otimes_R N$, or the tensor product of $M$ and $N$ over $R$. If the ring $R$ is clear from context we may omit $R$. The tensor product $M \otimes N$ is the quotient of the free abelian group $M \times N$ by the subgroup generated by

\[
\begin{align*}
(a + b, c) &= (a, c) + (b, c) \\
(a, b + c) &= (a, b) + (a, c)
\end{align*}
\]

The image of $(m, n)$ is denoted $m \otimes n$. $M \otimes N$ is an $R$-module via

\[
r(m \otimes n) = (rm) \otimes n = m \otimes (rn).
\]

Let $R$ be a subring of a ring $S$. (Then $S$ is an $R$-module in a natural way.) $S \otimes_R R$ is an $S$-module via

\[
s \left( \sum_{i=1}^{k} s_i \otimes r_i \right) = \sum_{i=1}^{k} ss_i \otimes r_i
\]

**Lemma 0.4.** The map $S \otimes_R R \to S$ is well defined and is an isomorphism of $S$-modules.

$s \otimes r \mapsto rs$

**Proof.** Consider the map $\Theta : S \times R \to S$, $(s, r) \mapsto sr$. This is $R$-bilinear. This gives the map $S \otimes_R R \to S$ above.

Consider the map $\phi : S \to S \otimes_R R$, $s \mapsto s \otimes 1$. 

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Then:
\[(\phi \circ \Theta)(\sum_{i=1}^k s_i \otimes r_i) = \phi(\sum_{i=1}^k s_ir_i) = (\sum_{i=1}^k s_ir_i) \otimes 1 = \sum_{i=1}^k s_i \otimes r_i\]

\[(\Theta \circ \phi)(s) = \Theta(s \otimes 1) = s. \text{ So this completes the lemma.}\]

**Corollary of HW:** If \(R\) is a subring of a ring \(S\) then \(R^n \otimes_R S \cong (R \otimes_R S)^n \cong S^n\) via \((r_1, \ldots, r_n) \otimes s \mapsto (r_1 \otimes s, \ldots, r_n \otimes s) \mapsto (r_1s, \ldots, r_ns)\)

Eg. If \(d\) is a field and \(V\) is a vector space over \(k\) and \(L\) is a field containing \(k\) then \(V \cong k^n, V \otimes_k L \cong L^n. \text{ The } n \text{-dimensional vectorspace over } L. \text{ If } v_1, \ldots, v_n \text{ is a basis for } V \text{ over } k \text{ then } v_1 \otimes 1, \ldots, v_n \otimes 1 \text{ is a basis for } V \otimes_k L \text{ over } L. \text{ This operation is called changing the base of } V \text{ from } k \text{ to } L.\)

**Definition 0.5.** If \(A, B\) are algebras over \(R\) (a ring) then in particular they are modules over \(R\). Then \(A \otimes_R B\) is an \(R\)-module and can be made into an \(R\)-algebra via \((a \otimes b)(a' \otimes b') = (aa') \otimes (bb')\) and extending this definition \(R\)-linearly.

Claim: This is well defined:

**Proof.** (sketch): We want a map \(A \otimes B \times A \times B \to A \otimes B\). Consider \(A \times B \times A \times B \to A \otimes B, a,b,a',b' \mapsto (aa') \otimes (bb'). \text{ Check: This is } R\text{-bilinear. Therefore we get a map: } (A \otimes B) \otimes (A \otimes B) \to A \otimes B. \text{ We know that this comes from:}\)

\[
\begin{array}{ccc}
M \times N & \to & M \otimes_R N \\
\downarrow \ & & \downarrow \\
& L & \\
\end{array}
\]

\((A \otimes B) \times (A \otimes B) \to A \otimes B \text{ which takes } (a \otimes b, a' \otimes b') \mapsto aa' \otimes bb' \square\)

Fact: \(A \otimes_R B\) with maps \(A \to A \otimes B, a \mapsto a \otimes 1 \text{ and } B \to A \otimes B,)
Let $b \mapsto 1 \otimes b$ satisfy the universal property:

\[
\begin{array}{c}
A \\
| \ \downarrow f \downarrow \\
| \ \downarrow g \\
A \otimes B \\
| \ \downarrow b \mapsto 1 \otimes b \\
B
\end{array}
\]

Also the fibered product exists for affine schemes. $\text{Spec } A \otimes A \otimes B = \text{Spec}(A \otimes_R B)$ eg.

We know that $\mathbb{A}^n_k \leftrightarrow \text{Spec } k[x_1, \ldots, x_n]$. What is $\mathbb{A}^n_k \times \mathbb{A}^m_k$ as a scheme.

$\mathbb{A}^n_k \times \mathbb{A}^m_k \leftrightarrow \text{Spec } k[x_1, \ldots, x_n, y_1, \ldots, y_m] \leftrightarrow \mathbb{A}^{m+n}_k$

Warning: The Zariski topology on $\mathbb{A}^{n+m}$ is not the product topology. For general schemes we glue the constructions above. Thm 3.3 in Hartshorne. For any two schemes $x, y$ over a base $S$ the fibered product $X \times_S Y$ exists.