Algebraic Geometry I
Lectures 22 and 23

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1 Fibered Products (contd..)

Recall the definition of fibered products

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\exists!\theta"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r] \arrow[d] & X \arrow[d, "\pi_1"'] \\
& Y \arrow[r, "\pi_2"] & S
\end{tikzcd}$$

By the universal mapping property of fibered products,

$$(X \times_S Y)(Z) = \{(f, g) \in X(Z) \times Y(Z) | \pi_1 \circ f = \pi_2 \circ g\}$$

For example, consider $Z = \text{Spec } k$ for some field k. Let X', Y', S' be affine varieties over k and let X, Y, S be the corresponding schemes over k i.e $X = \text{Spec } A(X')$ etc. (Note that if k is algebraically closed, then $X(k) = X'$).
Then $(X \times_S Y)(k) = \{(x, y) \in X(k) \times Y(k) | \pi_1(x) = \pi_2(y) \text{ in } S(k)\}$
In fact, in the category of sets, we have

$$X \times_S Y = \{(x, y) \in X \times Y | \pi_1(x) = \pi_2(y) \text{ in S}\}$$

For example, if we consider $S = \text{Spec } (k)$ above, then

$$X \times_S Y = \{(x, y) \in X \times Y\}$$

The construction of the fibered product generalizes many other elementary ones:

(1) If S contains only one element, then it is just $X \times Y$

(2) If $X \hookrightarrow^{\pi_1} S$, $Y \hookrightarrow^{\pi_2} S$ are inclusions, then it is $X \cap Y$

(3) If $Y \hookrightarrow^{\pi_2} S$ is an inclusion, then we get $\pi_1^{-1}(Y)$

(4) If $X = Y$, it gives the set on which the maps π_1, π_2 are equal i.e the *equalizer* of π_1, π_2

We want an analogue of (3) for fibers of schemes over points. (In (3) above, if $Y = \{s\}$ then we get $\pi_1^{-1}(s)$ i.e the fiber of π_1 over s).
If S is a scheme and $s \in S$ (i.e a map $\{s\} \longrightarrow S$ of sets), we want to associate a scheme to s and a map from that scheme to S.

Definition 1.1. Let $x \in X$, where X is a scheme. Let $\mathcal{O}_x$ be the stalk of $\mathcal{O}_X$ at x and $\mathfrak{m}_x$ the unique maximal ideal of $\mathcal{O}_x$. We define the *residue field* of x on X as $k(x) = \mathcal{O}_x/\mathfrak{m}_x$.

We now claim that there exists a canonical isomorphism

$$i_x : \text{Spec } k(x) \longrightarrow X \text{ whose image is } x.$$

To prove this, we begin by noting that $\exists U \in X$ such that $x \in U$ and

$$\psi : (U, \mathcal{O}_X|_U) \longrightarrow (\text{Spec } A,\, \mathcal{O}_{\text{SpecA}})$$

is an isomorphism for some ring A. Let $\wp = \psi(x)$ be a prime ideal of A. Then ψ induces a local isomorphism $\mathcal{O}_x \longrightarrow A_\wp$ whence we have an isomorphism

$$\begin{tikzcd} k(x) = \mathcal{O}_x/\mathfrak{m}_x \arrow[r, "\backsimeq"] & A_\wp/\wp A_\wp \\ & A_\wp \arrow[u] \\ & A \arrow[u] \end{tikzcd}$$

Here, the vertical maps are inclusions. This in turn gives rise to

$$\begin{tikzcd}
\mathrm{Speck(x)} \arrow[r, "\cong"] & \mathrm{Spec}(A_\wp/\wp A_\wp) \arrow[d] \\
& \mathrm{Spec}A_\wp \arrow[d] \\
& \mathrm{Spec}A \arrow[d, "(\cong)(\psi^{-1})"'] \\
& U
\end{tikzcd}$$

whence we have

$$\begin{tikzcd}
(0) \arrow[r] \arrow[dddr, "i_x"] & (0) \arrow[d] \\
& \wp A_\wp \arrow[d] \\
& \wp \\
& x \arrow[u]
\end{tikzcd}$$

Thus we see that

$$i_x^\sharp(s) = \begin{cases} \text{pullback of the image of } \ s \ \text{ in } \ \mathcal{O}_x & \text{if } \ x \in U \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.2. Let $f : X \longrightarrow Y$ be a morphism of schemes and let $y \in Y$. We thus have the following commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "\phi"] \arrow[d, "\psi"] & X \arrow[d, "f"] \\
\mathrm{Spec} \ \ k(y) \arrow[r, "i_y"] & Y
\end{tikzcd}$$

We define the *fiber* of f over y to be the scheme $X_y = X \times_Y \mathrm{Spec} \ \ k(y)$

Remark 1.3. If Z is a scheme, then

$$\begin{aligned} X_y(Z) &= (X \times_Y \mathrm{Spec}k(y))(Z) \\ &= \{(\phi, \psi) \in X(Z) \times (\mathrm{Spec} \ \ k(y))(Z) | f \circ \phi = i_y \circ \psi = y\} \end{aligned}$$

For example, if Z =Spec k and X,Y are varieties over k, then

$$X_y(k) = \{x \in X(k) | f(x) = y\}$$

So, as a set, $X_y(k)$ = fiber over y.

Example 1.4. See Ex 3.3.1 in Hartshorne.

Now consider the general situation

$$\begin{tikzcd} X \times_s \mathrm{Spec} k(s) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec} k(s) \arrow[r] & S \ni s \end{tikzcd}$$

Example 1.5. Consider the map

$$\begin{tikzcd} S = \mathrm{Spec}\mathbb{R}[t] \\ X = \mathrm{Spec}\mathbb{R}[x,y,t]/(x^2 - y^2 - t) \arrow[u] \end{tikzcd}$$

obtained from

$$\begin{tikzcd} \mathbb{R}[t] \arrow[d] \\ \mathbb{R}[x,y,t]/(x^2 - y^2 - t) \end{tikzcd}$$

Geometrically, f is the projection from $x^2 - y^2 = t$ to the t^{th} co-ordinate. For example, if $t = a$, then $f^{-1}(a) = \{(x,y) | x^2 - y^2 = a\}$. For $t = a = 0$, $f^{-1}(0) = \{(x,y) | x^2 - y^2 = 0\}$. The point $t = a$ corresponds to the maximal ideal $s = (t - a) \in S$.

$$\begin{aligned} X \times_s k(s) &= \mathrm{Spec}(\mathbb{R}[x,y,t]/(x^2 - y^2 - t)) \otimes (\mathbb{R}[t]/(t-a)) \\ &= \mathrm{Spec}\mathbb{R}[x,y,t]/(x^2 - y^2 - t, t - a) \\ &= \mathrm{Spec}\mathbb{R}[x,y]/(x^2 - y^2 - a) \end{aligned}$$

If $a \neq 0$, then this is irreducible. If $a = 0$, it is reducible.

1.1 Some Arithmetic Examples

Example 1.6. Spec $\mathbb{Z}$
The elements of Spec $\mathbb{Z}$ are the prime ideals of $\mathbb{Z}$ i.e (0), (p) for prime p. The closed subsets are of the form

$$\begin{aligned} V((n)) &= \{\text{prime ideals containing}(n)\} \\ &= \begin{cases} \text{primes dividing } \ n & \text{if } \ n \neq 0 \\ \text{Spec } \ \mathbb{Z} & \text{if } \ n = 0 \end{cases} \end{aligned}$$

So the closure of (0) is Spec $\mathbb{Z}$ i.e (0) is a *generic* point.

Example 1.7. Consider the map

$$\begin{array}{c} X = \operatorname{Spec}(\mathbb{Z}[x,y]/(x^2 - y^2 - 5)) \\ \Big\downarrow f \\ S = \operatorname{Spec}\mathbb{Z} \end{array}$$

obtained from

$$\begin{array}{c} \mathbb{Z}[x,y]/(x^2 - y^2 - 5) \\ \Big\uparrow \\ \mathbb{Z} \end{array}$$

Any prime p corresponds to $(p) \in S$.
Residue field $= \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} =$ quotient field of $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z}$.

$$\begin{aligned} \text{Fiber over } \ (p) &= \operatorname{Spec}(\mathbb{Z}[x,y]/(x^2 - y^2 - 5) \otimes \mathbb{Z}/p\mathbb{Z}) \\ &= \operatorname{Spec}(\mathbb{Z}[x,y]/(x^2 - y^2 - 5, p)) \\ &= \operatorname{Spec}(\mathbb{Z}/p\mathbb{Z}[x,y]/(x^2 - y^2 - 5)) \end{aligned}$$

So the fiber is the curve $(x^2 - y^2 - 5)$ in $\mathbb{A}^2_{\mathbb{F}_p}$.

Thus we see that schemes unite algebra and geometry (of polynomials). Earlier, polynomial equations were thought of geometrically while now we think of fibers algebraically. The example above also shows that one should think of the map obtained from $X \longrightarrow S$ as a family of schemes parametrized by points of S. In particular, a scheme over Spec $\mathbb{Z}$ is a family of curves over $\mathbb{Q}$ and $\mathbb{F}_p$ for primes p. This allows us to pass from char 0 objects to objects in char p. This process is an example of *specialization.*The fiber over (0) is called the *generic fiber* and that over (p) is called the *special fiber* over p. This motivates the following definition.

Definition 1.8. In the diagram

$$\begin{array}{ccc} X \times_S T & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

$X \times_S T$ is called the *base change* of X from S to T.

Example 1.9. (Geometric)
Consider the map

$$\begin{array}{c} \mathrm{Spec}\mathbb{C}[t][x]/(x^2 - t) \\ \downarrow \\ \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\mathbb{C}[t] \end{array}$$

Fiber over $a \in \mathbb{A}^1_{\mathbb{C}}$ is Spec $\mathbb{C}[x]/(x^2 - a)$. If $a \neq 0$, then $x^2 = a$ has 2 solutions and it has 1 solution if $a = 0$.

Example 1.10. (Number Theoretic)
Let $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ i.e the ring of integers in $\mathbb{Q}[i]$.
Consider

$$\begin{array}{c} \mathrm{Spec}(\mathbb{Z}[i]) \\ \downarrow \\ \mathrm{Spec}\mathbb{Z} \end{array}$$

obtained from

$$\begin{array}{c} \mathbb{Z}[i] \\ \uparrow \\ \mathbb{Z} \end{array}$$

If $p \in \mathbb{Z}$ is a prime, then the fiber over $(p) \in$ Spec $\mathbb{Z}$ is

$$\mathrm{Spec}(\mathbb{Z}[i] \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})) = \mathrm{Spec}(\mathbb{Z}[i]/p\mathbb{Z}[i])$$

So elements of the fiber are prime ideals of $\mathbb{Z}[i]$ containing p.

Thus schemes give a geometric picture of the behavior of primes in ring extensions of $\mathbb{Z}$. Schemes therefore incorporate "algebraic number theory" into "algebraic geometry".