1 Proj S

**Definition 1.1.** If \((a)\) is a homogeneous ideal of \(S\) then we define \(V((a)) = \{P \in \text{Proj}(S) | (a) \supseteq P\}\)

**Lemma 1.2.** (a) If \((a)\) and \((b)\) are homogeneous ideals of \(S\), then \(V((a) \cdot (b)) = V((a)) \cup V((b))\)
(b) If \(\{a_i\}_{i \in I}\) is a family of homogeneous ideals of \(S\) then \(V(\sum a_i) = \cap V(a_i)\)

**Proof.** Same as for spectra, using the fact that a homogeneous ideal \(P\) is prime iff whenever \(a, s \in S\) are homogeneous elements such that \(ab \in P\) then \(a \in P\) or \(b \in P\). \(\square\)

**Remark 1.3.** \(\text{Proj}(S) = V(0)\) and \(\phi = V(S)\)

**Definition 1.4.** The Zariski topology on \(\text{Proj}(S)\) is the one with closed subsets being \(V((a))\) for homogeneous ideals \((a)\).

**Example 1.5.** If \(S = k[x_1, \ldots, x_n]\) with \(k\) algebraically closed, then the set of closed points of \(\text{Proj}(S)\) (with the induced topology) is homeomorphic to \(\mathbb{P}^n_k\).

**Example 1.6.** \(n=2 (x - z, y - 2z)\) is a homogeneous prime ideal and corresponds to \([1:2:1]\) in \(P^2_k\). Also \(k[x, y, z]/(x - y, y - 2z) \cong k[z]\) with points in \(P^2_k\) are lines through the origin in \(A^3_k\).

Next: Define a sheaf of rings on \(\text{Proj}(S)\) If \(P \in \text{Proj}(S)\); let \(T_P\) be the set of all homogeneous elements \(S \setminus P\) and let \(S(P)\) denote the degree zero element in \(T_P^{-1}S\) and \(S(P)\) will be the local ring at \(P\).

**Example 1.7.** \(S = k[x, z]\), the point \(P=(0,1) \leftrightarrow P=(\text{homogeneous polynomials that do not vanish at } [0:1])\) then \(x^2 + y^2 \in T_P x^2 + xz = x(x + z) \in P\) and \(\frac{xz}{x^2 + y^2} \in S(P)\)
Definition 1.8. If \( v \in \text{Proj}(S) \) is open, then
\[
O(U) = \{ \text{functions} : U \to \bigsqcup S((p)| \forall p \in U, \exists \text{ a nbhd. } V \text{ of } P \text{ and homogeneous elements } a, f \in S \text{ of the same degree such that } \forall a \in V, f \notin (a) \text{ and } S((a)) = \frac{a}{f} \text{ in } S(y) \}.
\]

Example 1.9. This defines a sheaf, \( O((a), O_{\text{Proj}(S)} \)

Proposition 1.10. Let \( S \) be a graded ring.
(a) \( \forall p \in \text{Proj}(S), O_p \cong S_p(P) \) (stalk ), a local ring
(b) \( \forall \text{ homogeneous } f \in S, \text{ let } D_t(f) = \{ p \in \text{Proj}(S) | f \notin P \} = \text{Proj}(S) \setminus V((f)) \).

Proof. See book. \( \square \)

Example 1.11. Again if \( S = k[x, y, z] \) and \( f = z \), then the closed points of \( D_t(z) \) correspond to points \([x,y,z]\) in \( P_k^2 \) with \( z \neq 0 \)
i.e. these are the points of the type \([x:y:1]\), a copy of \( A^2 \)
Then \( D_t(f) \) is open in \( \text{Proj}(S) \) and the \( D_t(f) \)'s cover \( \text{Proj}(S) \), and there is an isomorphism of locally ringed spaces \( (D_t(f), O|_{D_t(f)}) \cong \text{Spec}(S(f)) \text{ where } S(f) \text{ is the subring of degree zero elements in } S_f \).

Remark 1.12. If \( k \) is algebraically closed field, then \( P_k^n \) may denote the corresponding scheme of variety.

For pictures of \( P_k^n \) see Mumford or Eisenbud-Harris.

Remark 1.13. The scheme \( P_k^n \) is the right place to do intersection theory and there is a Bezout’s theorem(See E-H).

Remark 1.14. Varieties first defined \( A^n_k \) and \( P^n_k \) and we considered closed subsets \( Z(a) \)
but for Affine Schemes: \( \text{Spec}(k[x_1, \cdots, x_n]) \) and \( \text{Spec}(k[x_1, \cdots, x_n])/(a) \) for Projective Schemes: \( \text{Proj}(k[x_1, \cdots, x_n]) \leftrightarrow P^n_k \) and \( \text{Proj}(k[x_1, \cdots, x_n]/(a)) \leftrightarrow Z((a)) \).