1 Varieties

Proposition 1.1. (a) If $X$ and $Y$ are algebraic sets in $\mathbb{A}^n$, then so is $X \cup Y$

(b) If $I$ is a set and $\forall i \in I, X_i$ is an algebraic set in $\mathbb{A}^n$, then $\bigcap_{i \in I} X_i$ is an algebraic set.

(c) $\phi$ and $\mathbb{A}^n$ are algebraic sets.

Proof. (a) Let $X = Z(T_1)$ and $Y = Z(T_2)$, where $T_1$ and $T_2$ are subsets in $k[x_1, \ldots, x_n]$. Then $X \cup Y = Z(T_1 T_2) = \{f_1 f_2 | f_1 \in T_1, f_2 \in T_2\}$ whence $X \cup Y$ is an algebraic set.

(b) Let $X_i = Z(T_i) \forall i \in I$. Then $\bigcap_{i \in I} X_i = Z(\bigcup_{i \in I} T_i)$.

(c) Note that $\phi = Z(1)$ and $\mathbb{A}^n = Z(0)$.

So the algebraic sets are the closed sets of a topology, called the Zariski topology. In other words, in the Zariski topology on $\mathbb{A}^n$, the closed sets are precisely the algebraic subsets.

Example 1.2. The complement of $y = x^2$ in $\mathbb{A}^n$ is open. Thus we see that open sets are big!
Example 1.3. Consider the Zariski topology on $\mathbb{A}^1$. Note that $k[x_1] = k[x]$ is a PID. So the algebraic sets are of the form $Z(f)$ for some $f(x) \in k[x]$. If $f(x) = c(x - a_1) \cdots (x - a_n)$, then $Z(f) = \{(a_1, \ldots, a_n)\}$ which is a finite set of points unless $n = 0$. If $n = 0$, then

$$Z(f) = \begin{cases} \emptyset & \text{if } c \neq 0 \\ \mathbb{A}^n & \text{if } c = 0 \end{cases}$$

Conversely, if $a_1, \ldots, a_n \in k$ with $n \geq 1$ then $\{(a_1), \ldots, (a_n)\} = Z((x - a_1), \ldots, (x - a_n))$. Also, as shown earlier, $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$. Thus the closed sets of $\mathbb{A}^1$ are either $\emptyset$, $\mathbb{A}^n$ or a finite set of points. In particular, the topology is not Hausdorff.

Now let $Y \subseteq \mathbb{A}^n$ be a subset (not necessarily algebraic). If $f(\overline{x}) \in k[\overline{x}]$ such that $f(\overline{a}) = 0 \forall \overline{a} \in Y$ then $fg = 0$ on $Y$ for any $g \in k[\overline{x}]$. Also if $f = 0$ and $g = 0$ on $Y$ then $f + g = 0$ on $Y$. Thus $\{f \in k[\overline{x}]|f = 0 \text{ on } Y\}$ is an ideal of $k[\overline{x}]$. It is denoted by $I(Y)$ and is called the ideal of $Y$.

**Proposition 1.4.**  
(a) If $T_1 \subseteq T_2$ are subsets of $k[\overline{x}]$ then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$ then $I(Y_1) \supseteq I(Y_2)$.

*Proof.* Homework. \(\square\)

Now if $a$ is an ideal of $k[\overline{x}]$, then a natural question to ask is: how is $I(Z(a))$ related to $a$? Certainly $a \subseteq I(Z(a))$. Does the reverse inclusion hold as well?

**Example 1.5.** Let $a = (x^n) \subseteq k[x]$ for some $n > 1$. Then $Z(a) = \{(0)\}$. Now observe that $x \in I(Z(a))$ but $x \not\in a$.

Thus $I(Z(a)) \nsubseteq \emptyset$.

**Definition 1.6.** If $a$ is an ideal in a ring $A$ then the radical of $a$ is defined as

$$\sqrt{a} = \{a \in A|a^n \in a; n \geq 1\}$$

An ideal $a$ is said to be a radical ideal if $a = \sqrt{a}$.

(HW): Show that for any ideal $a$, $\sqrt{a}$ is an ideal and is a radical ideal.
Remark 1.7. $\forall$ ideals $a$, $a \subseteq \sqrt{a}$. If $a$ is an ideal in $k[\bar{x}]$, then $\sqrt{a} \subseteq I(Z(a))$.

**Theorem 1.8.** (Hilbert’s Nullstellensatz) Assume $k$ is algebraically closed. Let $a$ be an ideal of $k[\bar{x}]$ and let $f \in k[\bar{x}]$ such that $f = 0$ on $Z(a)$. Then $f^r \in a$ for some $r$.

*Proof.* See textbook. \qed

**Corollary 1.9.** If $a$ is an ideal of $k[\bar{x}]$ then $I(Z(a)) = \sqrt{a}$

Remark 1.10. The conclusion of Theorem 1.8 need not hold if $k$ is not algebraically closed.

**Example 1.11.** Consider $k = \mathbb{R}$, $n = 2$ and $a = (x^2 + y^2 + 1) \subseteq k[x, y]$. Then $Z(a) = \phi$

and $I(Z(a)) = I(\phi) = k[x, y] \not\supseteq a$

**Example 1.12.** Consider $k = \mathbb{R}$, $n = 1$, $a = (x^3 - 1) \subseteq k[x]$. Then $Z(a) = \{(1)\} \subseteq \mathbb{A}^1_{\mathbb{R}}$

and $I(Z(a)) = (x - 1) \not\supseteq \sqrt{(x^3 - 1)} = \sqrt{a}$

**Proposition 1.13.** If $Y$ is any subset of $\mathbb{A}^n$ then $Z(I(Y)) = \overline{Y}$, where $\overline{Y}$ is the closure of $Y$ in the Zariski topology.

**Corollary 1.14.** There is a one-to-one inclusion reversing correspondence:

$$\{\text{radical ideals of } k[x_1, \ldots, x_n]\} \longleftrightarrow \{\text{closed subsets of } \mathbb{A}^n\}$$

given by

$$a \mapsto Z(a)$$

and

$$Y \mapsto I(Y)$$

We now introduce the concept of irreducible sets in $\mathbb{A}^n$. 

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Consider \( f(x, y) = x^2 - y^2 \in k[x, y] \) and \( Z(f) \subseteq A^2 \). We see that
\[
Z(f) = Z(x + y) \cup Z(x - y)
\]
so that \( Z(f) \) is the union two proper algebraic (i.e closed) subsets.

**Definition 1.15.** A non-empty subset \( Y \) of the topological space \( X \) is said to be *irreducible* if it cannot be expressed as a union of two proper closed subsets (they need not be disjoint).

Note that the empty set is considered to be *not irreducible*.

**Example 1.16.** \( Z(x^2 - y^2) \subseteq A^2 \) is not irreducible.

Recall that a topological space \( X \) is said to be *connected* if it cannot be written as a union of disjoint non-empty open subsets.

**Remark 1.17.** Consider \( Y = Z(x^2 - y^2) \). We claim that \( Y \) is connected in the Zariski topology.

**Lemma 1.18.** Irreducible \( \implies \) connected

**Proof.** Suppose \( Y \) is not connected i.e \( \exists \) non-empty open subsets \( U_1, U_2 \) such that \( U_1 \cup U_2 = Y \) and \( U_1 \cap U_2 = \emptyset \). We can then write
\[
Y = (Y \setminus U_1) \cup (Y \setminus U_2)
\]

Irreducibility is not as relevant in the "usual topology" as it is in algebraic geometry.

**Corollary 1.19.** Every algebraic set in \( A^n \) can be expressed uniquely as a union of irreducible subsets with not one containing any other.

**Proof.** See textbook.

**Definition 1.20.** An *affine algebraic variety* is an irreducible closed subset of \( A^n \) with the induced (Zariski) topology.

**Proposition 1.21.** If \( Y_1, Y_2 \) are subsets of \( A^n \), then
\[
I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)
\]
Corollary 1.22. An algebraic set $Y$ in $\mathbb{A}^n$ is irreducible $\iff I(Y)$ is a prime ideal.

Proof. ($\implies$) Let $Y$ be irreducible and suppose that $f, g \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Thus $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both being closed subsets of $Y$. However, $Y$ is irreducible so that we must have either $Y = Y \cap Z(f)$ whence $Y \subseteq Z(f)$, or $Y = Y \cap Z(g)$ whence $Y \subseteq Z(g)$. Thus either $f \in I(Y)$ or $g \in I(Y)$.

($\impliedby$) Suppose $p = I(Y)$ is prime. Then $Y = Z(p) = Z(I(Y))$. Suppose $Z(p) = Y_1 \cup Y_2$ with $Y_1, Y_2$ closed subsets. Then $p = I(Z(p)) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ whence $p = I(Y_1)$ or $p = I(Y_2)$. Thus $Z(p) = Y_1$ or $Y_2$, hence it is irreducible.

Example 1.23. $\mathbb{A}^n$ is irreducible since $I(\mathbb{A}^n) = (0)$ which is prime

Example 1.24. If $f \in k[x_1, \ldots, x_n]$ is irreducible then $(f)$ is a prime ideal and so $Z(f)$ is irreducible. For $n = 2$, $Z(f)$ is called a curve. For $n = 3$, $Z(f)$ is called a surface. For $n > 3$, $Z(f)$ is called a hypersurface.

HW* A non-empty open subset of an irreducible space is irreducible and dense.

Non-empty open subsets of an irreducible space carry a lot of information about the topology.

Definition 1.25. A non-empty open subset of an affine variety is called a quasi-affine variety.

Lemma 1.26. If $Y = Y_1 \cup Y_2$ with $Y_1$, $Y_2$ connected and $Y_1 \cap Y_2 \neq \emptyset$, then $Y$ is connected.

Proof. Suppose $Y$ is not connected so that $\exists$ open subsets $U_1$, $U_2$ with $Y = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Consider a point $P \in (Y_1 \cap Y_2)$. Then either $P \in Y_1$ or $P \in Y_2$. WLOG, suppose that $P \in Y_1$. Now note that

$$Y_1 = (Y_1 \cap U_1) \cup (Y_1 \cap U_2)$$

Since $Y_1$ is connected, one of $Y_1 \cap U_1$ and $Y_1 \cap U_2$ must be empty. But we know that $P \in Y_1 \cap U_1$ and so $Y_1 \cap U_2$ is empty. Thus $Y_1 \subseteq U_1$. Similarly, $Y_2 \subseteq U_1$ and thus $Y \subseteq U_1$ whence $U_2 = \emptyset$. ($\Rightarrow \Leftarrow$)

Example 1.27. It immediately follows from the above lemma that $Z(x^2 - y^2)$ is connected in $\mathbb{A}^2$ since both $Z(x - y)$ and $Z(x + y)$ are connected (they are both irreducible).