Algebraic K-Theory and $\zeta(s)$

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Zeta functions

Part 1: Cohomological approach to the Riemann Hypothesis

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Generalization: If R is a commutative ring,

$$\zeta_R(s) = \prod_{\mathfrak{m} \in \operatorname{MaxSpec} R} \frac{1}{1 - |R/\mathfrak{m}|^{-s}}$$

$\zeta(s) = \zeta_{\mathbb{Z}}(s)$

Zeta functions

Theorem (Riemann Hypothesis over a finite field)

If X is a curve over $\mathbb{F}_q$, then $\zeta_X(s) = 0$ implies $Re\, s = \frac{1}{2}$.

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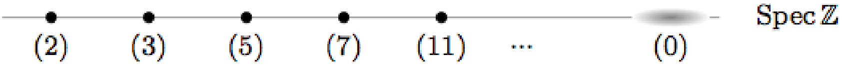
Deninger ('90s): Use a Weil cohomology theory in *Arakelov geometry* to prove the Riemann Hypothesis.

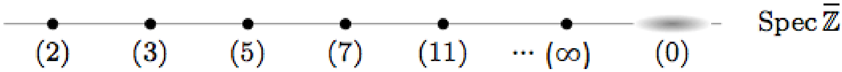
Arakelov geometry – algebraic geometry over $\operatorname{Spec} \bar{\mathbb{Z}}$, the hypothetical one-point compactification of $\operatorname{Spec} \mathbb{Z}$.

Arakelov geometry

Arakelov geometry – algebraic geometry over Spec $\bar{\mathbb{Z}}$, the hypothetical one-point compactification of Spec $\mathbb{Z}$.



Algebraic geometry: point $=$ prime ideal in $\mathbb{Z}$



Algebraic number theory: point $=$ valuation on $\mathbb{Q}$

$v_p(n) = \#$powers of p dividing n

$v_\infty(n) = \ln|n|$

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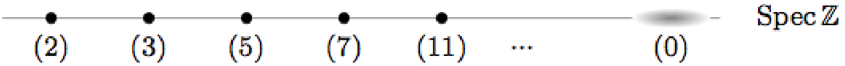

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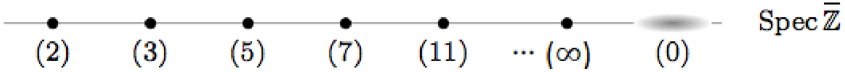

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$\operatorname{Spec}\bar{\mathbb{Z}}$ itself is not a real scheme, but

Definition (Durov (2009))

$\bar{\mathbb{Z}}$-module $=$ abelian group $+$ norm on $A \otimes \mathbb{R}$.

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1. $\hat{\zeta}(s) = \zeta_\infty(s) \prod_p \frac{1}{1-p^{-s}}$ where $\zeta_\infty(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$

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2. Tate Duality (1962, class field theory): $H^i(\mathbb{Z}) \cong H_c^{3-i}(\mathbb{Z})$ suggests that
   - compactification $\mathrm{Spec}\, \mathbb{Z}$ = closed 3-manifold
   - compare Poincare Duality: $H^i(M) \cong H_c^{n-i}(M)$ if M^+ is a closed n-manifold

See also: *Knots and Primes* -Morishita

Summary so far:

1. $\mathrm{Mod}_{\bar{\mathbb{Z}}}$ – category of 'normed abelian groups'

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(if H^* Morita invariant, $\mathrm{Mod}_R \simeq \mathrm{Mod}_S$ implies $H^*(R) \simeq H^*(S)$)

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Question: What are examples of cohomology theories $H^*(R)$ which can be defined in terms of Mod_R? (*Morita invariance*)

- **Morita invariant**: **algebraic K-theory**, Hochschild homology, cyclic homology, periodic cyclic homology
- **not Morita invariant**: **etale cohomology**, crystalline cohomology, any known Weil cohomology theory

Two big problems:

1. Construct a Morita invariant Weil cohomology theory.
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Part 2: Algebraic K-theory

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1. $GL(R)$ = group of infinite invertible matrices with entries in R, equal to I except in finitely many places

$$\begin{pmatrix} [M] & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}, \; M \in \mathrm{GL}_n(R)$$

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$K_*\mathbb{Z}$ is very hard to compute but related to number theory:

Example

(Kurihara 1992) $K_{4n}\mathbb{Z} = 0 \leftrightarrow$ Vandiver Conjecture (ca 1850!);

Algebraic K-theory

$K_0(\mathbb{Z}) = \mathbb{Z}$	$K_8(\mathbb{Z}) = (0?)$	$K_{16}(\mathbb{Z}) = (0?)$	$K_{8k}(\mathbb{Z}) = (0?),\ k \geq 1$
$K_1(\mathbb{Z}) = \mathbb{Z}/2$	$K_9(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{17}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{8k+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$
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Example (Main Conjecture, Iwasawa Theory)

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$$\frac{|K_{2i-2}X| \cdot |K_{2i-3}\mathbb{F}_q|}{|K_{2i-1}\mathbb{F}_q| \cdot |K_{2i-3}X|} = |\zeta_X(1-i)|$$

Conjecture

$$\frac{|K_{4k-2}\bar{\mathbb{Z}}|}{|K_{4k-1}\bar{\mathbb{Z}}|} = |\hat{\zeta}(1-2k)| \text{ exactly}$$

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$\mathcal{C} = \mathrm{Mod}_R$, $\mathrm{Ext}^i(X, Y) = \mathrm{RHom}^i(X, Y)$ *right derived functors*
$K_*\mathrm{Mod}_R \cong K_*R$

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Theorem (Localization Theorem a la Thomason 1980s)

If $\mathcal{C} \subseteq \mathcal{D}$, there is a long exact sequence

$$\cdots \to K_n(\mathcal{C}) \to K_n(\mathcal{D}) \to K_n(\mathcal{D}/\mathcal{C}) \to K_{n-1}(\mathcal{C}) \to \cdots$$

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Example

$$\cdots \to \bigoplus_p K_n\mathbb{F}_p \to K_n\mathbb{Z} \to K_n\mathbb{Q} \to \bigoplus_p K_{n-1}\mathbb{F}_p \to \cdots$$

even though there is no $R \to \mathbb{Z} \to \mathbb{Q}$ with $K_nR \cong \bigoplus_p K_n\mathbb{F}_p$.

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- related by: $K_n R = \pi_n(KR)$

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Philosophy: Spectra are only defined up to homotopy equivalence.

Definition (1960s)

A **spectrum** (a la algebraic topology) is a:

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Definition (2000s, Joyal/Lurie)

An **∞-category** is a category in which morphisms are only defined up to homotopy, objects up to homotopy equivalence.

(no notion of isomorphic objects or equal morphisms)

Examples of ∞-categories:

- spaces up to homotopy equivalence ('homotopy types' or '∞-groupoids')
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- purely algebraic techniques of *higher algebra* (Lurie/Nikolaus/Mathew/Scholze/etc., 2010s)

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Example

If $\mathcal{R}$ = abelian groups, an $\mathcal{R}$-enriched category is a category $\mathcal{C}$ for which morphisms $f, g : X \to Y$ can be added and subtracted.
Example: Mod_R is enriched in abelian groups via $\mathrm{Hom}_R(M, N)$
- or in spectra via $\mathrm{Ext}^*_R(M, N)$

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Solution: $\mathrm{Mod}_{\bar{\mathbb{Z}}}$ is enriched in 'Arakelov spectra'!
- spectrum E + norms on $E \otimes \mathbb{R}$

My work:

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Find a Morita invariant Weil cohomology theory

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$\rightarrow$ Riemann Hypothesis

Enriched ∞-categories

Theorem (B. 2020[1])

$EnrCat_{\mathcal{R}}$ is a full subcategory of $Mod^{\star}_{\mathcal{R}}$.

- *$Mod^{\star}_{\mathcal{R}} = \mathcal{R}$-modules with set of distinguished objects*
- *$\mathcal{R}$-module $\mathcal{M}$ with set S of objects in the essential image if and only if $\mathcal{M} \to \mathcal{R}^S$ is a conservative functor of $\mathcal{R}$-modules*

(enriched ∞-categories $\subseteq$ higher algebra)

Example

$\mathcal{S} = \infty$-category of spectra
$\mathcal{S}$-enriched ∞-categories $\subseteq$ $\mathcal{S}$-module
$K_* - \mathcal{S}$-module invariant with universal property

[1] Enriched ∞-categories I: enriched presheaves. https://arxiv.org/abs/2008.11323

K_* of an enriched ∞-category

K-theory is defined on

$$\{\text{rings/schemes}\} \subseteq \{\text{spectrum-enriched } \infty\text{-categories}\} \subseteq \{\mathcal{S}\text{-modules}\}$$

Using this **algebraic model** for $\mathcal{R}$-enriched categories,

Definition (universal property for K-theory)

K-theory is the universal invariant of $\mathcal{R}$-modules

$$K : \mathrm{Mod}^{\star}_{\mathcal{R}} \to \mathcal{R}$$

sending Verdier quotients to cofibers (Blumberg-Gepner-Tabuada)

Questions:

- Are these definitions equivalent?
- $K_*(\mathrm{Spec}\,\bar{\mathbb{Z}})$ is now well-defined! How do we compute it?