Enumerative Geometry of Pascal's Theorem

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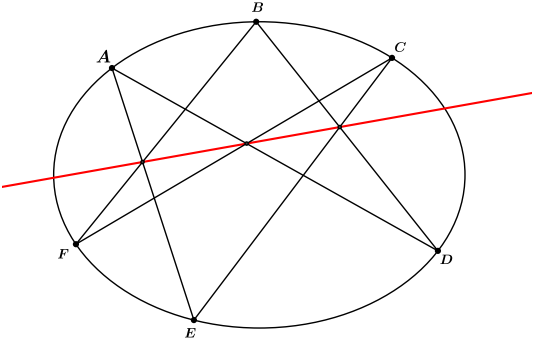
Pascal's Theorem



The Pascal line is denoted by $\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}$.

The same six points give several different lines such as

$$\left\{ \begin{matrix} B & A & F \\ C & D & E \end{matrix} \right\}, \qquad \left\{ \begin{matrix} F & A & C \\ D & E & B \end{matrix} \right\}, \quad \text{etc.}$$

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Altogether we get $\frac{6!}{2\times 3!} = 60$ Pascal lines.

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Can we go backwards?

6 points on a conic

$\Rightarrow$ both depend on 6 parameters

3 lines in a plane

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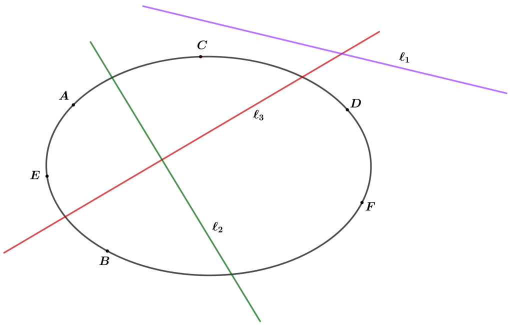
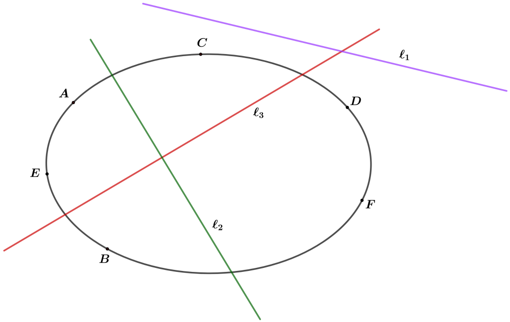
Can we pre-specify 3 Pascal lines and look for the original 6 points?



C
ℓ_1
A
D
ℓ_3
E
F
ℓ_2
B



Can we find points $(A, B, \ldots, F)$ such that

$$\left\{\begin{matrix} A & B & C \\ F & E & D \end{matrix}\right\}, \quad \left\{\begin{matrix} F & A & C \\ D & E & B \end{matrix}\right\}, \quad \left\{\begin{matrix} E & B & C \\ F & D & A \end{matrix}\right\}$$

respectively correspond to green, red and purple lines?

We expect a FINITE number of solutions.

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The Computation

Fix the conic: $xz = y^2$.

Fix an isomorphism

$$\mathbb{P}^1 \xrightarrow{\sim} \text{conic}, \qquad a \longrightarrow [1 : a : a^2] = A.$$

Then the Pascal line $\left\{\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right\}$ has homogeneous coordinates

$$u_x = abde - abdf + \cdots + bcdf,$$
$$u_y = abf - abe + \cdots + cde,$$
$$u_z = ae - ad + \cdots + cf.$$

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It is equal the line $\ell = [\ell_x : \ell_y : \ell_z]$ exactly when the 2×2 minors of the matrix

$$\begin{bmatrix} u_x & u_y & u_z \\ \ell_x & \ell_y & \ell_z \end{bmatrix}$$

are zero.

In this way, we get three ideals

$$I_1,\ I_2,\ I_3 \subseteq \mathbb{C}[a,b,c,d,e,f].$$

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Now find the **degree** of J, which is the required number.

There are 77 cases modulo the action of the symmetric group on $\{A, B, \ldots, F\}$.

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For example,

$$\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}, \left\{ \begin{matrix} A & B & D \\ E & C & F \end{matrix} \right\}, \left\{ \begin{matrix} A & B & D \\ C & E & F \end{matrix} \right\} \rightsquigarrow \mathbf{14}$$

Galois covers

Motivating case:

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \left\{ \begin{array}{ccc} A & B & D \\ F & E & C \end{array} \right\}, \left\{ \begin{array}{ccc} A & E & D \\ F & B & C \end{array} \right\} \rightsquigarrow \mathbf{2}$$

We can find a *formula* for the two solutions $(A, B, \ldots, F)$.

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Problem: In the remaining cases, find the **Galois group** of the corresponding cover.

The Steiner case

We have

$$\left\{\begin{matrix} A & B & C \\ F & E & D \end{matrix}\right\}, \left\{\begin{matrix} A & B & C \\ E & D & F \end{matrix}\right\}, \left\{\begin{matrix} A & B & C \\ D & F & E \end{matrix}\right\} \rightsquigarrow \mathbf{0}$$

because the Pascal lines are concurrent.

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Fix three concurrent lines ℓ_1, ℓ_2, ℓ_3. Then the solutions $(A, B, \ldots, F)$ move on a curve X such that

$$X \longrightarrow \mathbb{P}^1$$

is a branched cover of degree 7.

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What is the geometry of X?

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- Blow it up to get a morphism

$$\Sigma \longrightarrow \mathbb{P}^2.$$

- Find the intersection ring of Σ.

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Thank You!