Solution to § 5.2: 3(a)

Solve: $y'' - xy' - y = 0$, $x_0 = 0$ — (i)

Sol: Try $y = \sum_{n=0}^{\infty} a_n x^n$

Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Putting this in (i), we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Make all exponents of $t$ become $n$:

$$\sum_{n=2}^{\infty} \frac{(n+3)(n+2)}{2} a_{n+2} x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Make all sums start at same index by separating some initial terms:

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) a_{n+2} - n a_n - a_0 \right] x^n = 0$$

Equating coefficients of various powers of $x$, we get

$2a_2 - 2a_0 = 0$ and for $n = 1, 2, 3, \ldots$

$$a_n + \frac{a_0}{2} (n+1) = \frac{a_n}{n+2}$$

$$a_0 = \frac{a_0}{2}$$

$$a_n = \frac{a_n}{n+2}$$