Solution to §5.2: 3(a)

Solve: $y'' - xy' - y = 0$, $x_0 = 0$ —— (i)

Sol: Try $y = \sum_{n=0}^{\infty} a_n x^n$

Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

& $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Putting this in (i), we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\therefore \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

make all exponents of t become n:

$$\sum_{n=2-2}^{\infty} (n+2)((n+2)-1) a_{n+2} x^{n+2-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

ie $$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

make all sums start at the same index by separating some initial terms:

$$(0+2)(0+1) a_{0+2} x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - a_0 x^0 - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$\therefore 2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n\right] x^n = 0$$

Equating coefficients of various powers of x, we get

$2a_2 - a_0 = 0$ & for $n = 1, 2, 3, \ldots$

$(n+2)(n+1) a_{n+2} - n a_n - a_n = 0$

ie $a_2 = \frac{a_0}{2}$

ie $a_{n+2} = \frac{n a_n + a_n}{(n+2)(n+1)} = \frac{\cancel{(n+1)} a_n}{(n+2)\cancel{(n+1)}} = \frac{a_n}{n+2}$