Computing intersection numbers between abelian varieties associated to subspaces of modular forms

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Abstract

We state a result and describe an algorithm for computing the intersection number between abelian varieties associated to complementary subspaces of the space of cuspidal modular forms.

1 Introduction

Let $N > 5$ to be an integer. If g is a newform of level N_g dividing N, then let $[g]$ denote its Galois conjugacy class. Denote by $S_{([g],\mathbb{C})}$ the subspace in $S_2(\Gamma_0(N),\mathbb{C})$ generated by degeneracy map images of forms in $[g]$. Denote by $S_{([g],\mathbb{Q})}$ the $\mathbb{Q}$-vector space of all such forms with rational Fourier coefficients. Then $S_2(\Gamma_0(N),\mathbb{Q}) = \oplus_{[g]} S_{([g],\mathbb{Q})}$ where the sum is over the Galois conjugacy classes of newforms at levels that divide N. Let $X_0(N)$ denote the modular curve over $\mathbb{Q}$ associated to $\Gamma_0(N)$, and let $J_0(N)$ be its Jacobian, which is an abelian variety over $\mathbb{Q}$. Let $\mathbf{T}$ denote the subring of endomorphisms of $J_0(N)$ generated by the Hecke operators (usually denoted T_ℓ for $\ell \nmid N$ and U_p for $p \mid N$). If $X = \oplus_{[g]} S_{([g],\mathbb{C})}$ and $Y = \oplus_{[f]} S_{([f],\mathbb{C})}$ are two subpaces of $S_2(\Gamma_0(N),\mathbb{C})$ with trivial intersection, then the *intersection number between X and Y* is the order of intersection between the two associated abelian varieties $(J_0(N)/\mathrm{Ann}_{\mathbf{T}}(X)J_0(N))^\vee$ and $(J_0(N)/\mathrm{Ann}_{\mathbf{T}}(Y)J_0(N))^\vee$, where the superscript $\vee$ denotes the dual abelian variety, and the intersection is taken inside $J_0(N)^\vee$ (which is isomorphic to $J_0(N)$).

One reason for studying intersection numbers between such spaces X and Y is as a means to study congruence primes between X and Y, since [ARS12, Thm. 3.6] shows that the primes that divide the intersection number are congruence primes, and a congruence prime whose square does not divide the level N divides the intersection number. Another reason for studying the intersection number is that it

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plays a role in the second part of the Birch and Swinnerton-Dyer conjecture: for example, if the level N is prime and X is associated to a single newform, then it is shown in [Aga10] (see equation (2) and Lemma 4.4 in loc. cit.) that a certain intersection number divides the Birch and Swinnerton-Dyer conjectural order of the Shafarevich-Tate group of the abelian variety associated to the newform.

We now prepare to state the main theorem of this article. Let g denote the genus of $X_0(N)$. Given a basis $B = \{s_1, \ldots, s_{2g}\}$ for $S_2(\Gamma_0(N), \mathbb{C})$, the matrix for the Hecke operator t relative to the basis B is the unique matrix M that satisfies

$$t(s_i) = \sum_{j=1}^{2g} M_{ij}(s_j).$$

If M is a matrix with rational entries, then let d denote the least common multiple of all of the denominators of entries of M. We denote by M_{int} the matrix given by $d \cdot M$, the integer matrix produced by clearing denominators in M. If M is an $m \times n$ matrix with integer coefficients, we define the *torsion order* of M to be the product of the diagonal entries of the Smith Normal Form of M. We denote the torsion order of M as $\text{tors}(M)$. The torsion order of M is so called because it is the order of the torsion part of the group $\mathbb{Z}^m/\text{Col}(M)$ where $\text{Col}(M)$ the denotes subgroup of $\mathbb{Z}^m$ generated by the columns of M. Given a set of $m \times n$ matrices, $\{M_1, \ldots, M_k\}$, we denote by $M_1 \ldots M_k$ the $m \times (nk)$ matrix produced by the horizontal augmentation of the $\{M_i\}$.

We have the following result.

Theorem 1. *Let $X = \oplus_{[g]} S_{([g],\mathbb{C})}$ and $Y = \oplus_{[f]} S_{([f],\mathbb{C})}$, where the sums are over two complementary subsets of the set of Galois conjugacy classes of newforms at levels that divide N. Let $\{t_1, \ldots, t_m\}$ and $\{r_1, \ldots, r_m\}$ be generators for the annihilator ideals in $\boldsymbol{T} \otimes \mathbb{Q}$ of X and Y respectively. Let $\{T_1, \ldots, T_m\}$ and $\{R_1, \ldots, R_n\}$ be matrices for $\{t_1, \ldots, t_m\}$ and $\{r_1, \ldots r_n\}$ respectively relative to some basis of $S_2(\Gamma_0(N))$. Then the intersection number between X and Y is*

$$\frac{\text{tors}(T_{1_{\text{int}}} \ldots T_{m_{\text{int}}} R_{1_{\text{int}}} \ldots R_{n_{\text{int}}})}{\text{tors}(T_{1_{\text{int}}} \ldots T_{m_{\text{int}}}) \cdot \text{tors}(R_{1_{\text{int}}} \ldots R_{n_{\text{int}}})} \tag{1.1}$$

In Section 2, we present an algorithm for computing intersection numbers based on the theorem above, and in section 3, we give the proof of Theorem 1.

We remark that the key idea in this paper is that the annihilator ideals in the theorem above and in the algorithm can be computed over $\mathbb{Q}$. The ideals are more naturally defined over $\mathbb{Z}$ and there is a simpler formula analogous to the one in the theorem above over $\mathbb{Z}$; however doing linear algebra over the integers takes too long. Replacing the ideals over $\mathbb{Z}$ by their tensor over $\mathbb{Q}$ speeds up the process significantly.

2 The Algorithm

As usual, instead of working with $S_2(\Gamma_0(N))$, we work with the dual space $\mathrm{H}_1(X_0(N),\mathbb{Z})\otimes\mathbb{R}$, which in turn can be described by cuspidal modular symbols. The computations can be done conveniently using the mathematical software SAGE.

Denote the subspaces of cuspidal modular symbols corresponding to X and Y by X_H and Y_H respectively; in fact, in actual applications, usually the spaces X_H and Y_H are specified instead of X and Y. Let $I_{(X,\mathbb{Q})}$ and $I_{(Y,\mathbb{Q})}$ denote the annihilator ideals in $\mathbf{T}\otimes\mathbb{Q}$ of X_H and Y_H respectively. Theorem 1 applies mutatis mutandis to the space of cuspidal modular symbols instead of $S_2(\Gamma_0(N))$, and indicates an obvious algorithm to compute the intersection number between X and Y using modular symbols, provided one can compute generators in $\mathbf{T}\otimes\mathbb{Q}$ for $I_{(X,\mathbb{Q})}$ and $I_{(Y,\mathbb{Q})}$. So it suffices to indicate how to find these generators efficiently in practise, which is what we do next.

Let $s_1,\ldots,s_{2g}$ be a basis for the modular symbols space for $\Gamma_0(N)$. We compute a basis for $\mathbf{T}\otimes\mathbb{Q}$, say $\{t_1,\ldots,t_g\}$ by searching for a Hecke operator t with distinct eigenvalues and setting $t_i=t^i$ (this notation for t_i differs from the notation used in Theorem 1, and will be used only in this section). We remark that in our discussion, the Hecke matrices are $2g\times 2g$. However, one can exploit the complex conjugation involution on $H_1(X_0(N),\mathbb{Z})$ and only work with $g\times g$ matrices. This is what we do in practice. In SAGE, the command X.modular_symbols() computes a basis for X_H where the basis elements are expressed as linear combinations of $s_1,\ldots,s_{2g}$. Suppose the basis consists of k elements, given by $\sum_{j=1}^{2g} r_{ij}s_j$ for $i=1,\ldots,k$. Let M be the $k\times 2g$ matrix whose (i,j)-th entry is r_{ij}. Then M gives a surjection $\mathrm{H}_1(X_0(N),\mathbb{Z})\to X_H$, which we denote by $\widetilde{M}$. For an $m\times n$ matrix A, denote by vector(A) the row vector of $m\cdot n$ entries constructed by horizontally augmenting the rows of A. Denote the matrices associated to the basis $\{t_1,\ldots,t_g\}$ of $\mathbf{T}\otimes\mathbb{Q}$

by $\{T_1,\ldots,T_g\}$. Then for some weights c_i,

$$
\begin{aligned}
& (\sum_{i=1}^{g} c_i t_i)|_X = 0|_X \\
\iff \quad & \sum_{i=1}^{g} \widetilde{M}(c_i t_i) = 0 \\
\iff \quad & \sum_{i=1}^{g} c_i \widetilde{M} t_i = 0 \\
\iff \quad & \begin{bmatrix} c_1 \\ . \\ . \\ . \\ c_g \end{bmatrix} \in ker\left(\begin{bmatrix} \text{vector}(MT_1) \\ . \\ . \\ . \\ \text{vector}(MT_g) \end{bmatrix}^{\mathrm{T}} \right).
\end{aligned} \tag{2.1}
$$

Thus a set of generators of $I_{(X,\mathbb{Q})}$ are the matrices $\sum_{i=1}^{g} c_i T_i$ as $(c_1,\ldots,c_g)$ range over a basis for the kernel mentioned in equation (2.1).

Our initial goal in coming up with the algorithm in this article was that the existing command intersection_number() in sage was too slow to compute intere-section numbers between a given newform and the complementary space when the level N was above 1000. Unfortunately, our new algorithm was is no faster for this purpose. However, our algorithm is useful in other contexts where the annihilator ideals I_X and I_Y in $\mathbf{T}$ of X and Y are specified in some other way. For example, in [Aga99] and [Aga00, Chap 3], the ideal I_X is specified as the annihilator under the action of $\mathbf{T}$ of a particular element of $H_1(X_0(N),\mathbb{Z}) \otimes \mathbb{R}$ called the winding element. The command intersection_number() needs one to specify the actual sub-spaces X and Y rather than their annihilator ideals. Thus our algorithm will be useful when the ideals are specified rather than the subspaces. Indeed, a version of the algorithm was used by the first author in [Aga00, §3.3], for the quotient of J by the annihilator of the winding element (called the winding quotient) when the level N is prime, but the details were skipped in loc. cit. (the results were also reported in [Aga99]). Replacing the annihilator ideal over $\mathbb{Z}$ by its tensor over $\mathbb{Q}$, and thus doing linear algebra over $\mathbb{Q}$ as opposed to over $\mathbb{Z}$ allowed us to do com-putations faster and at higher levels in loc. cit. We hope that our algorithm will be useful in other similar contexts. In fact, another context where the first author plans to use it is in the situation of [Aga11] which has a formula for the special L-value of the winding quotient of level a product of two distinct primes that involves the intersection number (this calculation is relevant for the Birch and Swinnerton-Dyer conjecture for this quotient).

3 Proof of Theorem 1

If X is a subspace of $S_2(\Gamma_0(N))$ of the form $\oplus_{[g]} S_{([g],\mathbb{C})}$, where the sum is over a subset of the set of Galois conjugacy classes of newforms at levels that divide N, then denote by I_X the annihilator ideal of $\oplus_{[g]} S_{([g],\mathbb{Q})}$ in the Hecke algebra $\mathbf{T}$. Recall that $I_{(X,\mathbb{Q})}$ denotes the annihilator ideal of $\oplus_{[g]} S_{([g],\mathbb{Q})}$ in $\mathbf{T} \otimes \mathbb{Q}$.

Lemma 1. *Let $X = \oplus_{[g]} S_{([g],\mathbb{C})}$ and $Y = \oplus_{[f]} S_{([f],\mathbb{C})}$, where the sums are over two complementary subsets of the set of Galois conjugacy classes of newforms at levels that divide N. Then the intersection number between X and Y is the order of the group*

$$\frac{\mathrm{H}_1(X_0(N),\mathbb{Z})}{\mathrm{H}_1(X_0(N),\mathbb{Z})[I_X] + \mathrm{H}_1(X_0(N),\mathbb{Z})[I_Y]} \tag{3.1}$$

Proof. This follows from Lemmas 4.1 and 4.3 of [Aga10]. □

Lemma 1 is the basis for our computation. However, directly finding generators for I_X and I_Y is computationally difficult, since one has to solve equations with integer coefficients. The remaining lemmas below show how to modify Lemma 1 so that one may instead use generators for $I_{(X,\mathbb{Q})}$ and $I_{(Y,\mathbb{Q})}$, which are less costly to compute since one can work over the rational numbers.

For later reference, we now establish some simple facts regarding group orders; these facts are well known, and we give proofs only for the sake of completeness.

Lemma 2. *Let G be a finitely generated abelian group, and let H_1 and H_2 be subgroups of G. If $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$ are subgroups of finite index in H_1 and H_2 respectively, then*

$$\left|\frac{H_1 + H_2}{K_1 + K_2}\right| = \frac{|H_1/K_1| \cdot |H_2/K_2|}{|(H_1 \cap H_2)/(K_1 \cap K_2)|}$$

Proof. Consider the following commutative diagram of exact sequences.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 \cap K_2 & \longrightarrow & K_1 \times K_2 & \longrightarrow & K_1 + K_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1 \cap H_2 & \longrightarrow & H_1 \times H_2 & \longrightarrow & H_1 + H_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \frac{H_1 \cap H_2}{K_1 \cap K_2} & & \frac{H_1 \times H_2}{K_1 \times K_2} & & \frac{H_1 + H_2}{K_1 + K_2} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The Nine Lemma completes the bottom row so that

$$0 \longrightarrow \tfrac{H_1 \cap H_2}{K_1 \cap K_2} \longrightarrow \tfrac{H_1 \times H_2}{K_1 \times K_2} \longrightarrow \tfrac{H_1 + H_2}{K_1 + K_2} \longrightarrow 0$$

is exact. In particular

$$\frac{H_1 + H_2}{K_1 + K_2} \cong \frac{(H_1 \times H_2)/(K_1 \times K_2)}{(H_1 \cap H_2)/(K_1 \cap K_2)}.$$

The result follows. $\square$

For a finitely generated abelian group G, denote the order of the torsion subgroup of G as $\operatorname{tors}(G)$.

Lemma 3. *Let G be a finitely generated abelian group, and let H_1 and H_2 be subgroups of G such that G/H_1 and G/H_2 are both torsion-free and $\frac{G}{H_1+H_2}$ is finite. If $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$ are subgroups of finite index in H_1 and H_2 respectively, then*

$$\left|\frac{G}{H_1 + H_2}\right| = \left|\frac{G}{K_1 + K_2}\right| \cdot \frac{\operatorname{tors}(G/(K_1 \cap K_2))}{\operatorname{tors}(G/K_1) \cdot \operatorname{tors}(G/K_2)}$$

Proof. By the third Isomorphism Theorem we have

$$\left|\frac{G}{H_1 + H_2}\right| = \frac{\left|\frac{G}{K_1+K_2}\right|}{\left|\frac{H_1+H_2}{K_1+K_2}\right|} \tag{3.2}$$

However, by Lemma 2 above,

$$\left|\frac{H_1 + H_2}{K_1 + K_2}\right| = \frac{|H_1/K_1| \cdot |H_2/K_2|}{|(H_1 \cap H_2)/(K_1 \cap K_2)|}$$

so that equation (3.2) becomes

$$\left|\frac{G}{H_1+H_2}\right| = \frac{|G/(K_1+K_2)|\cdot|(H_1\cap H_2)/(K_1\cap K_2)|}{|H_1/K_1|\cdot|H_2/K_2|}. \tag{3.3}$$

By the third Isomorphism Theorem, for $i=1$ and $i=2$ we have

$$G/H_i \cong \frac{G/K_i}{H_i/K_i} \tag{3.4}$$

as well as

$$G/(H_1\cap H_2) \cong \frac{G/(K_1\cap K_2)}{(H_1\cap H_2)/(K_1\cap K_2)} \tag{3.5}$$

Since, G/H_i are torsion-free, equation (3.4) says that $|H_i/K_i| = \mathrm{tors}(G/K_i)$. Likewise, equation (3.5) indicates $|(H_1\cap H_2)/(K_1\cap K_2)| = \mathrm{tors}(G/(K_1\cap K_2)$. So equation (3.3) becomes

$$\left|\frac{G}{H_1+H_2}\right| = \frac{|G/(K_1+K_2)|\cdot \mathrm{tors}(G/(K_1\cap K_2))}{\mathrm{tors}(G/K_1)\cdot\mathrm{tors}(G/K_2)}$$

as desired. □

Similar to the notation for matrices defined in the introduction, if I is an ideal in $\mathbf{T}\otimes\mathbb{Q}$ with generators $\{t_1,\ldots,t_n\}$, we will write I_{int} to denote the ideal in $\mathbf{T}$ generated by $\{z_1t_1,\ldots,z_nt_n\}$, where $z_i\in\mathbb{Z}$ is the smallest positive integer such that $z_it_i\in\mathbf{T}$. In this case, we will also denote z_it_i by $t_{i_{int}}$. (Note that I_{int} depends on the choice of generators $\{t_i\}$ and on the integers $\{z_i\}$, so there does not exist a one-to-one correspondance between ideals in $\mathbf{T}\otimes\mathbb{Q}$ and ideals in $\mathbf{T}$ of the form I_{int}. However, we supress this fact in the following because this dependance does effect the results as stated.)

Lemma 4. *Let $X=\oplus_{[g]}S_{([g],\mathbb{C})}$ and $Y=\oplus_{[f]}S_{([f],\mathbb{C})}$, where the sums are over two complementary subsets of the set of Galois conjugacy classes of newforms at levels that divide N. Then $I_{(X,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})$ is a subgroup of finite index in $\mathrm{H}_1(X_0(N),\mathbb{Z})[I_Y]$ and $I_{(Y,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})$ is a subgroup of finite index in $\mathrm{H}_1(X_0(N),\mathbb{Z})[I_X]$. The order of the group in equation (3.1) is equal to*

$$\frac{\left|\frac{\mathrm{H}_1(X_0(N),\mathbb{Z})}{I_{(Y,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})+I_{(X,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})}\right|}{\mathrm{tors}\left(\frac{\mathrm{H}_1(X_0(N),\mathbb{Z})}{I_{(X,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})}\right)\cdot\mathrm{tors}\left(\frac{\mathrm{H}_1(X_0(N),\mathbb{Z})}{I_{(Y,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})}\right)} \tag{3.6}$$

Proof. It is obvious that $I_{(X,\mathbb{Q})}(\mathrm{H}_1(X_0(N),\mathbb{Z})\otimes\mathbb{Q}$ is isomorphic to $(\mathrm{H}_1(X_0(N),\mathbb{Z})\otimes\mathbb{Q})[I_{(Y,\mathbb{Q})}]$, since they are both just Y_H. It is not difficult to see that this induces an isomorphism between $I_X\mathrm{H}_1(X_0(N),\mathbb{Z})\otimes\mathbb{Q}$ and $\mathrm{H}_1(X_0(N),\mathbb{Z})[I_Y]\otimes\mathbb{Q}$. Since the two spaces are equal upon tensoring with $\mathbb{Q}$, and $I_X\mathrm{H}_1(X_0(N),\mathbb{Z})\subseteq\mathrm{H}_1(X_0(N),\mathbb{Z})[I_Y]$, $I_X\mathrm{H}_1(X_0(N),\mathbb{Z})$ is a subgroup of finite index in $\mathrm{H}_1(X_0(N),\mathbb{Z})[I_Y]$. Likewise, $I_Y\mathrm{H}_1(X_0(N),\mathbb{Z})$ is a subgroup of finite index in $\mathrm{H}_1(X_0(N),\mathbb{Z})[I_X]$. The result follows from Lemma 3. $\square$

We will now show how the quantities in the numerator and denominator of equation (3.6) are equal to those occurring in equation (1.1), which will prove Theorem 1. Let $B=\{s_1,\dots s_{2g}\}$ be a basis for $\mathrm{H}_1(X_0(N),\mathbb{Z})$. Let $\{t_1,\dots,t_m\}$ and $\{r_1,\dots,r_n\}$ be generators $I_{(X,\mathbb{Q})}$ and $I_{(Y,\mathbb{Q})}$ respectively. Let $\{T_1,\dots,T_m\}$ and $\{R_1,\dots,R_n\}$ be matrices for $\{t_1,\dots,t_n\}$ and $\{r_1,\dots r_m\}$ relative to B.
The group $\frac{\mathrm{H}_1(X_0(N),\mathbb{Z})}{I_{(X,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})}$ has a presentation

$$\langle s_1,\dots,s_{2g}|t_{i_{int}}s_j=0 \text{ for } i=1,\dots,m, j=1,\dots,2g\rangle. \tag{3.7}$$

The group $\frac{\mathrm{H}_1(X_0(N),\mathbb{Z})}{I_{(Y,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})}$ has a presentation

$$\langle s_1,\dots,s_{2g}|r_{i_{int}}s_j=0 \text{ for } i=1,\dots,n, j=1,\dots,2g\rangle. \tag{3.8}$$

The group $\frac{\mathrm{H}_1(X_0(N),\mathbb{Z})}{I_{(Y,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})+I_{(X,\mathbb{Q})_{\mathrm{int}}}\mathrm{H}_1(X_0(N),\mathbb{Z})}$ has a presentation

$$\langle s_1,\dots,s_{2g}|t_{i_{int}}s_j=0 \text{ for } i=1,\dots,m, j=1,\dots,2g, r_{i_{int}}(s_j) \text{ for } i=1,\dots,n, j=1,\dots,2g\rangle. \tag{3.9}$$

Let e_i denote the i-th standard basis vector (with n entries, a 1 in the i-th position and 0's elswhere). Then using the map $s_i\mapsto e_i$, the group presented in equation (3.7) is isomorphic to $\mathbb{Z}^n/\mathrm{Col}(T_{1_{\mathrm{int}}}\dots T_{m_{\mathrm{int}}})$, the order of which is precisely $\mathrm{tors}(T_{1_{\mathrm{int}}}\dots T_{m_{\mathrm{int}}})$. Similarly, the groups in (2.2.14) and (2.2.15) can be computed as $\mathrm{tors}(T_{1_{\mathrm{int}}}\dots T_{m_{\mathrm{int}}}R_{1_{\mathrm{int}}}\dots R_{n_{\mathrm{int}}})$ and $\mathrm{tors}(R_{1_{\mathrm{int}}}\dots R_{n_{\mathrm{int}}})$ respectively. Theorem 1 follows.

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