A generalization of Kronecker’s first limit formula to GL(n)

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Abstract
Kronecker’s first limit formula gives the polar and constant terms of the Laurent series expansion of the Eisenstein series for SL(2, \mathbb{Z}) at \( s = 1 \). In this article, we generalize the formula to certain maximal parabolic Eisenstein series associated to SL(\( n, \mathbb{Z} \)) for \( n \geq 2 \). We also show how the generalized formula can be used to express the polar and constant terms of the Dedekind zeta function of any number field at \( s = 1 \).

1 Introduction
Let \( n \geq 2 \) be an integer. Let \( \tau \) be in the generalized upper half-plane \( \mathcal{H}^n \), which consists of \( n \times n \) matrices with real number entries that are the product of an upper triangular matrix with 1’s along the diagonal and a diagonal matrix with positive diagonal entries such that the lowermost diagonal entry is 1. When \( n = 2 \), one can identify \( \mathcal{H}^n \) with the usual complex upper half plane. For details, see, e.g., [Gol06, §1.2].

In the following, \( m_1, \ldots, m_n \) denote integers with perhaps some added restrictions as noted; in particular, we follow the convention that in any sum over a subset of \( m_1, \ldots, m_n \), if a term has denominator zero for some values of \( m_1, \ldots, m_n \), then the term is to be skipped in the sum.

Consider the maximal parabolic Eisenstein series

\[
E_n(\tau, s) = \sum_{(m_1, \ldots, m_n) = 1} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n) \tau \|^s ns/2},
\]

where \( \| (m_1 \ldots m_n) \tau \| \) denotes the norm of the row vector that is the product of the row vector \( (m_1 \ldots m_n) \) and the matrix \( \tau \); this series converges when \( \Re(s) > 1 \), and is known to have a meromorphic continuation to all of \( \mathbb{C} \).

Let

\[
E^*_n(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \zeta(ns) E(\tau, s)
\]

\[
= \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n) \tau \|^s ns/2}.
\]

Note that in the case where \( n = 2 \), if \( \tau \) corresponds to the point \( z = x + iy \) in the complex upper half plane, then

\[
E^*_2(\tau, s) = \pi^{-s} \Gamma(s) \sum_{m_1, m_2} \frac{y^s}{|m_1 z + m_2|^s}.
\]

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The classical Kronecker’s first limit formula gives the first two terms of the Laurent expansion of $E_2^*(\tau, s)$ at $s = 1$: 

$$E_2^*(\tau, s) = \frac{1}{s-1} + (\gamma - \log 4\pi - \log y - 4\log |\eta(z)|) + O(s-1), \quad (2)$$

where $\eta(z)$ is the Dedekind eta-function. Note that sometimes Kronecker’s first limit formula is stated for Dedekind zeta functions or for Epstein zeta functions, but such formulas can be deduced from the formula above.

Kronecker’s first limit formula was generalized to the case $n = 3$ in [BG84] and [Efr92]. In this article, we generalize the formula to arbitrary $n \geq 2$ (see Theorem 1.1). A generalization of Kronecker’s first limit formula to arbitrary $n \geq 2$ is also given in [Ter73] for Epstein zeta functions.

**Theorem 1.1.** For a given $\tau \in \mathcal{H}^n$, let $y_1, \ldots, y_{n-1}$ denote the unique positive real numbers such that for $i \geq 1$, we have $\tau_{n-i,n-i} = \prod_{j=1}^{n-i} y_j$. If $m_1, \ldots, m_n$ are integers, then for $j = 2, \ldots, n$, let $b_j = \sum_{i=1}^{j-1} m_i \tau_{i,j}$ and $c_j = \tau_{j,j}$; also let $m$ be the nonnegative real number such that $m^2 = m_n^2/c_n^2 + m_{n-1}^2/c_{n-1}^2 + \cdots + m_2^2/c_2^2$ and $d = b_n m_n/c_n + b_{n-1} m_{n-1}/c_{n-1} + \cdots + 2 m_2/c_2$. Let $\tau'$ be the submatrix of $\tau$ obtained by removing the topmost row and leftmost column and let

$$g(\tau) = \exp \left\{ -\frac{1}{4} \left( \prod_{i=1}^{n-1} y_i \right) E_{n-1}^* \left( \tau', \frac{n}{n-1} \right) + \sum_{m_1 \neq 0} \frac{1}{m_1} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp \left( 2\pi i d - 2\pi |m_1|m \prod_{i=1}^{n-1} y_i \right) \right\}.$$ 

Then

$$E_n^*(\tau, s) = \frac{2/n}{s-1} + \left( \gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i \right) - 4\log g(\tau) \right) + O(s-1). \quad (3)$$

The theorem is proved in Section 3. The techniques used are elementary: one only needs some basic Calculus and the Poisson summation formula, which is recalled in equation (20). Our proof is a generalization of the proof of the classical Kronecker’s first limit formula given in [Lan87, §20.4], and the key observation is to perform the sum in (1) over $m_1, \ldots, m_n$ conveniently (we first sum over $\{m_1\}$ and then over $\{m_2, \ldots, m_n\}$ when $n = 2$, there is not much of a choice) and to apply the Poisson summation formula in the correct order (over $m_n$ first, followed by $m_{n-1}$, and so on, up to $m_2$). Even in the case $n = 3$, our proof differs in some key steps with that in [Efr92] (who introduces complex coordinates on $\mathcal{H}^3$, while we don’t) and that in [BG84] (who use minimal parabolic Eisenstein series). Our proof techniques are similar to those used in [Ter73] (about which we learned only after a first draft of this article was written), but are more direct and elementary (the main goal of [Ter73] is to prove the functional equation of the Epstein zeta function using generalizations of the Selberg-Chowla formula).

If we put $n = 2$ in the formula for $E_n^*(\tau, s)$ above, then we get the classical Kronecker’s first limit formula (2), with $g(\tau) = |\eta(\tau)|$ (this last equality follows from the correctness of our formula, but also because our proof is identical to the classical proof in the case $n = 2$ as given in [Lan87, §20.4]). Thus our function $g(\tau)$ for arbitrary $n$ is a generalization of $|\eta(\tau)|$. If we put $n = 3$, then we recover the formula for $E_3^*(\tau, s)$ given in [Efr92, Theorem 1].

We suspect that the expression for $E_n^*(\tau, s)$ above can be used to show that $g(\tau)$ is automorphic and that $\log(g(\tau))$ is a harmonic function on $\mathcal{H}^n$ (i.e., is annihilated by invariant differential operators), as is done for $n = 2$ (see, e.g., [Sie80, §I.2] for automorphy) and $n = 3$ (see [Efr92, §3]). This may be the approach to answer the question raised at the end of §1 and §3 in [Ter73]. However, we shall not pursue these issues in the present article.
The classical Kronecker limit formula has several applications (see, e.g., [Sie80]); many of these generalize to give applications of our generalization of the limit formula. In this article, we shall limit ourselves to one application: the classical limit formula for Eisenstein series can be used to give the polar and constant terms of the Laurent series expansion of the Dedekind zeta function (and of the partial zeta functions) at $s = 1$ of a quadratic imaginary field (due to Kronecker) and of a real quadratic field (due to Hecke) (for details, see [Sie80]). In Section 2, we give the analogous formulas for any number field (this was already done for cubic fields in [Efr92]): see Theorem 2.1.

2 Zeta functions of number fields

In this section, we a formula for the Laurent series expansion of the partial and Dedekind zeta function of a number field times some explicit functions; the main idea is to use a generalization of a trick of Hecke to express the partial zeta function as an integral of an Eisenstein series over a suitable region and then use our limit formula for the Eisenstein series. The procedure is described for $n = 3$ in [Efr92, §4], and the discussion below is its generalization. However, we do need a new input, which is formula (11), the analog of which was not required in loc. cit.

Let $K$ be a number field of degree $n$ over $\mathbb{Q}$. Let $r$ denote the number of real embeddings and $c$ denote the number of complex conjugate embeddings. We assume that $r + c > 1$ since the cases where $r + c = 1$ are classical (the field of rational numbers and quadratic imaginary fields).

Let $A$ be an ideal class of $K$. The partial zeta function associated to $A$ is given by

$$\zeta_K(s, A) = \sum_{a \in A} \frac{1}{N a^s}. \tag{4}$$

Fix $B \in A^{-1}$. Let $U$ denote the unit group of $K$ and let $w_K$ denote the number of roots of unity in $K$. Let $m = r + c - 1$, and let $\epsilon_1, \ldots, \epsilon_m$ denote a fundamental set of units. Then

$$\zeta_K(s, A) = \sum_{a \in A} \frac{1}{Na^s} = NB^s \sum_{\lambda \in B/U} \frac{1}{|N\lambda|^s} = \frac{NB^s}{w_K} \sum_{\lambda \in B/(\epsilon_1, \ldots, \epsilon_m)} \frac{1}{|N\lambda|^s}. \tag{5}$$

Order the embeddings of $K$ so that the first $c$ are complex and the remaining $r$ are real. If $x \in K$, then for $i = 1, \ldots, r + c$, let $|x|_i$ denote the absolute value of the image of $x$ under the $i$-th embedding. For $i = 1, \ldots, m + 1 = r + c$, let $\delta_i = 1$ if the $i$-th embedding is real and $\delta_i = 2$ otherwise.

We first deal with the case where $K$ has at least one real embedding. Then the $(m + 1)$-st embedding is real, and thus $\delta_{m+1} = 1$. For any positive real numbers $a_1, \ldots, a_{m+1}$, a change of variables shows that

$$\int_0^\infty \cdots \int_0^\infty (a_1^2 t_1^2 + \ldots + a_m t_m^2 + a_{m+1} (t_1 \cdots t_m)^{\delta_1 \cdots \delta_m - 2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} = (a_1^{\delta_1} \cdots a_{m+1}^{\delta_{m+1}})^{-s} \int_0^\infty \cdots \int_0^\infty (t_1^2 + \ldots + t_m^2 + (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}. \tag{6}$$

Now for $i = 1, \ldots, r + c$, let $a_i = |\lambda|_i$. Then $a_1^{\delta_1} \cdots a_{m+1}^{\delta_{m+1}} = |N\lambda|$. Let

$$d(s) = \int_0^\infty \cdots \int_0^\infty (t_1^2 + \ldots + t_m^2 + (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}. \tag{6}$$
Putting all this in formula (5), we see that

$$d(s) \frac{1}{|N\lambda|^s} = \int_0^\infty \cdots \int_0^\infty (|\lambda|^2 t_1^2 + \ldots + |\lambda|^2 t_m^2 + |\lambda|^2_{m+1}(t_1^{\delta_1} \ldots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m},$$

and so

$$d(s) \sum_{\lambda \in B \cap \langle \epsilon_1, \ldots, \epsilon_m \rangle} \frac{1}{|N\lambda|^s} \int_0^\infty \cdots \int_0^\infty (|\lambda|^2 t_1^2 + \ldots + |\lambda|^2 t_m^2 + |\lambda|^2_{m+1}(t_1^{\delta_1} \ldots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m},$$

Now $1 = |N\epsilon_i| = \prod_{j=1}^{m+1} |\epsilon_i|_{\delta_j}$, and so

$$|\epsilon_i|_{m+1} = (|\epsilon_i|_{\delta_1} \cdots |\epsilon_i|_{\delta_m})^{-1},$$

considering that $\delta_{m+1} = 1$. For $i = 1, \ldots, m$, let $\epsilon_i$ act on $(\mathbb{R}^+)^m$ by multiplying the $j$-th coordinate by $|\epsilon_i|_j$. Letting $D$ be a fundamental domain under the action of $\langle \epsilon_1, \ldots, \epsilon_m \rangle$ on $(\mathbb{R}^+)^m$, we get

$$\sum_{\lambda \in B \cap \langle \epsilon_1, \ldots, \epsilon_m \rangle} \int_0^\infty \cdots \int_0^\infty (|\lambda|^2 t_1^2 + \ldots + |\lambda|^2 t_m^2 + |\lambda|^2_{m+1}(t_1^{\delta_1} \ldots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}$$

$$= \sum_{\lambda \in B \cap D} (|\lambda|^2 t_1^2 + \ldots + |\lambda|^2 t_m^2 + |\lambda|^2_{m+1}(t_1^{\delta_1} \ldots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}$$

Let $\alpha_1, \ldots, \alpha_n \in K$ be a $\mathbb{Z}$-basis of $B$. Let $\lambda \in B$. Then $\lambda = m_1\alpha_1 + \cdots + m_n\alpha_n$ for some $m_1, \ldots, m_n \in \mathbb{Z}$. For $i = 1, \ldots, r + c$, let $\lambda_i$ denote the image of $\lambda$ under the $i$-th embedding. We denote $(\alpha_j)_{ij}$ by $\alpha_{ij}$. Recalling the way we ordered the embeddings of $K$, we see that for $i = 1, \ldots, c$, $|\lambda|^2_i = \Re(\lambda_i)^2 + \Im(\lambda_i)^2$, while for $i = c + 1, \ldots, c + r$, $|\lambda|^2_i = \lambda_i^2$ (if $c = 0$, then any expression below containing $\Re$ or $\Im$ should be ignored). Thus

$$|\lambda|^2 t_1^2 + \ldots + |\lambda|^2 t_m^2 + |\lambda|^2_{m+1}(t_1^{\delta_1} \ldots t_m^{\delta_m})^{-2}$$

$$= (m_1\Re(\alpha_{1,1}) + \ldots + m_n\Re(\alpha_{n,1}))^2 t_1^2 + (m_1\Im(\alpha_{1,1}) + \ldots + m_n\Im(\alpha_{n,1}))^2 t_1^2 + \ldots$$

$$+ (m_1\Re(\alpha_{1,c}) + \ldots + m_n\Re(\alpha_{n,c}))^2 t_c^2 + (m_1\Im(\alpha_{1,c}) + \ldots + m_n\Im(\alpha_{n,c}))^2 t_c^2$$

$$+ (m_1\alpha_{1,m+1} + \ldots + m_n\alpha_{n,m+1})^2 t_{m+1}^2 + \ldots$$

$$+ (m_1\alpha_{1,m+1} + \ldots + m_n\alpha_{n,m+1})^2 t_{m+1}^2$$

$$= \mathbf{m}M^T \mathbf{m}^T,$$

where $\mathbf{m}$ is the row vector with entries $m_1, \ldots, m_n$ and $M$ is the $n \times n$ matrix whose $i$-th row has entries

$$\Re(\alpha_{i,1}) t_1, \Im(\alpha_{i,1}) t_1, \ldots, \Re(\alpha_{i,c}) t_c, \Im(\alpha_{i,c}) t_c, \alpha_{i,c+1} t_{c+1}, \ldots, \alpha_{i,m} t_m, \alpha_{i,m+1}(t_1^{\delta_1} \ldots t_m^{\delta_m})^{-2}.$$
\[ \mathbb{R}^m \to \mathbb{S}^n \] (note that \( \det Q \) is independent of \( t_1, \ldots, t_m \)). Thus

\[
\sum_{\lambda \in \Lambda} \int_D \left( |\lambda|^2 t_1^2 + \ldots + |\lambda|^2 t_m^2 + |\lambda|^2 (t_1 \ldots t_m)^{2-2} \right)^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}
\]

\[
= \sum_{m_1, \ldots, m_n} \int_D (\mathbf{mMM}^T \mathbf{m}^T)^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}
\]

\[
= \sum_{m_1, \ldots, m_n} \int_D (\mathbf{mQm}^T)^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}
\]

\[
= \sum_{m_1, \ldots, m_n} \int_D (\mathbf{m} (\det Q)^{1/n} (\det \tau(t_1, \ldots, t_m))^{-2/n} \tau(t_1, \ldots, t_m) \tau(t_1, \ldots, t_m)^T \mathbf{m}^T)^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}
\]

\[
= (\det Q)^{-s/2} \int_D (\det \tau(t_1, \ldots, t_m))^s \sum_{m_1, \ldots, m_n} (\mathbf{m} \tau(t_1, \ldots, t_m) \tau(t_1, \ldots, t_m)^T \mathbf{m}^T)^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}
\]

\[
= (\det Q)^{-s/2} \pi^{ns/2} \Gamma(ns/2)^{-1} \int_D E_n^*(\tau(t_1, \ldots, t_m), s) \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m},
\]

by definition of \( E_n^*(\tau, s) \) (see formula (1)). Thus from the equation above and equations (4), (7), and (9), we see that

\[
d(s) \zeta_K(s, A) = NB^s \det Q^{-s/2} \pi^{ns/2} \Gamma(ns/2)^{-1} \int_D E_n^*(\tau(t_1, \ldots, t_m), s) \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}.
\]

Considering that for a complex number \( z \), \( \Re(z) = \frac{z + \overline{z}}{2} \) and \( \Im(z) = \frac{z - \overline{z}}{2} \), we see that

\[
4^c \det Q = \text{disc}B = (NB)^2 d_K, \quad \text{where} \quad d_K \text{ denotes the discriminant of } K.
\]

So

\[
d(s) \zeta_K(s, A) = \frac{(2^c d_K^{-1/2})^s}{w_K^n} \pi^{ns/2} \Gamma(ns/2)^{-1} \int_D E_n^*(\tau(t_1, \ldots, t_m), s) \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}. \quad (10)
\]

Now consider the case where \( K \) does not have a real embedding, i.e., \( K \) is totally complex. In that case, formula (5) does not work, and instead, we use the following formula:

\[
\int_0^\infty \cdots \int_0^\infty \left( a_1^2 t_1^2 + \ldots + a_m^2 t_m^2 + a_{m+1} (t_1 \ldots t_m)^{2-2} \right)^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}
\]

\[
= (a_1^2 \cdots a_{m+1}^2)^{-s} \int_0^\infty \cdots \int_0^\infty \left( t_1^2 + \ldots + t_m^2 + (t_1 \ldots t_m)^{2-2} \right)^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}, \quad (11)
\]

The argument above starting right after formula (5) goes through, with the following changes: take

\[
d(s) = \int_0^\infty \cdots \int_0^\infty (t_1^2 + \ldots + t_m^2 + (t_1 \ldots t_m)^{2-2})^{-ns/2} \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m}, \quad (12)
\]

replace the term \( (t_1^4 \cdots t_m^4) \) by \( (t_1 \cdots t_m) \), replace the \( \delta_i \)'s in equation (8) by 1's, and replace the last entry in the \( i \)-th row of \( M \) by the two entries \( \Re(\alpha_{i,m+1})(t_1 \cdots t_m)^{-2} \) and \( \Im(\alpha_{i,m+1})(t_1 \cdots t_m)^{-2} \). So equation (10) is still valid, but with \( d(s) \) given by formula (12), and with the definition of \( \tau(t_1, \ldots, t_m) \) modified as per the change in \( M \) mentioned above.

Let \( V = \int_D \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \). Then from equation (10) and our generalization of Kronecker's limit formula (formula (3)), we get
Theorem 2.1. Recall that $K$ is a number field that is not the rational numbers or a quadratic imaginary field. With notation as above, in particular, taking $d(s)$ to be given by formula (6) if $K$ has a real embedding, and by formula (12) if not, we have

$$w_K(2^{-c_n}/2d_K^{1/2})^s d(s) \Gamma(ns/2) \zeta_K(s, A) = \frac{2V/n}{s-1} + (\gamma - \log 4\pi)V - \frac{2}{n} \int_D \log \left( \prod_{i=1}^{n-1} y_i \right) \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m} - 4 \int_D \log g(\tau) \frac{dt_1}{t_1} \ldots \frac{dt_m}{t_m} + O(s - 1).$$

Finally, note that the Dedekind zeta function is the sum of the partial zeta functions over all ideal classes, so from the formula above, we get a corresponding formula involving the Dedekind zeta function.

3 Proof of Theorem 1.1

We first prove the formula for $E_n^*(\tau, s)$ and deduce from it the formula for $E_n^+\tau, s)$. In formula (1), the term corresponding to $m_1 = 0$ is

$$S_1 = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_2, \ldots, m_n} \frac{(\prod_{i=1}^{n-1} y_i^{n-i})^s}{\| (m_2 \ldots m_n) \tau' \|^n/2}$$

$$= \pi^{-ns/2} \Gamma(ns/2) \cdot \left( \prod_{i=1}^{n-1} y_i^{(n-i)/n} \right)^s \sum_{m_2, \ldots, m_n} \frac{(\prod_{i=1}^{n-2} y_i^{n-1-i})(n-1)}{(n-1)(\pi n s/2)}$$

$$= \left( \prod_{i=1}^{n-1} y_i^{(n-i)/n} \right)^s \cdot \pi^{-ns/2} \Gamma(ns/2) \cdot E_{n-1} \left( \tau', \frac{n}{n-1} s \right)$$

$$= \left( \prod_{i=1}^{n-1} y_i^{(n-i)/n} \right)^s \cdot E_{n-1}^* \left( \tau', \frac{n}{n-1} s \right). \quad (13)$$

Let

$$S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \ldots, m_n} \frac{\pi^{-ns/2} \Gamma(ns/2)}{\| (m_1 \ldots m_n) \tau \|^n/2},$$

so that

$$E_n^*(\tau, s) = S_1 + \left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^s \cdot S_2. \quad (14)$$

Our next goal is to find a suitable expression for $S_2$, which will not be achieved till equation (25) below. We use the formula

$$\frac{\pi^{-s} \Gamma(s)}{a^s} = \int_0^\infty \exp(-\pi at) t^s \frac{dt}{t} \quad (15)$$

with $a = \| (m_1 \ldots m_n) \tau \|$, and $s$ replaced by $ns/2$ to get

$$S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \ldots, m_n} \int_0^\infty \exp(-\pi t \| (m_1 \ldots m_n) \tau \|) t^{ns/2} \frac{dt}{t}. \quad (16)$$
For $j = 2, \ldots, n$, let
\[
a_j = (m_1 \tau_{1,1})^2 + \cdots + \left( \sum_{i=1,\ldots,j-1} m_i \tau_{i,j-1} \right)^2,
\]
and recall that $b_j = \sum_{i=1,\ldots,j-1} m_i \tau_{i,j}$, and $c_j = \tau_{j,j}$. Then
\[
\| (m_1 \ldots m_n) \tau \| = (m_1 \tau_{1,1})^2 + \cdots + \left( \sum_{i=1,\ldots,n-1} m_i \tau_{i,n-1} \right)^2 + \left( \sum_{i=1,\ldots,n-1} m_i \tau_{i,n} + m_n \tau_{n,n} \right)^2
\]
\begin{equation}
= a_n + (b_n + c_n m_n)^2,
\end{equation}
Putting (18) in (16), we get
\[
S_2 = \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \int_0^\infty \exp(-\pi t a_n) \exp(-\pi t(b_n + c_n m_n)^2) t^{n_s/2} \frac{dt}{t}
\]
\begin{equation}
= \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \int_0^\infty \exp(-\pi t a_n) \sum_{m_n} \exp(-\pi t(b_n + c_n m_n)^2) t^{n_s/2} \frac{dt}{t}
\end{equation}
The Poisson summation formula says that for real numbers $t, b, c$, with $c \neq 0$,
\[
\sum_{m \in \mathbb{Z}} \exp(-\pi t (b + cm)^2) = \frac{1}{c \sqrt{t}} \sum_{m \in \mathbb{Z}} \exp(2\pi ibm/c) \exp(-\pi m^2/ct^2).
\]
Using this with $b = b_n$ and $c = c_n$, replacing $m$ by $m_n$, and noting that $c_n$, being a diagonal entry of $\tau$, is always positive, we get
\[
\sum_{m_n} \exp(-\pi t(b_n + c_n m_n)^2) = \frac{1}{c_n \sqrt{t}} \exp(-\pi t a_n) \sum_{m_n} \exp(2\pi ib_n m_n/c_n) \exp(-\pi m_n^2/t c_n^2).
\]
Putting this in (19), we get
\[
S_2 = \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \int_0^\infty \exp(-\pi t a_n) \sum_{m_n} \exp(2\pi ib_n m_n/c_n) \exp(-\pi m_n^2/t c_n^2) t^{n_s/2-1/2} \frac{dt}{t}
\]
\begin{equation}
= \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \exp(2\pi ib_n m_n/c_n) \int_0^\infty \exp(-(\pi a_n t + \pi m_n^2/t c_n^2)) t^{n_s/2-1/2} \frac{dt}{t}
\end{equation}
\begin{equation}
= \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \exp(2\pi ib_n m_n/c_n) \cdot \int_0^\infty \exp(-\pi (m_n^2/c_n^2)/t) \sum_{m_2,\ldots,m_n-2} \sum_{m_{n-1}} \exp(-\pi a_n t) t^{n_s/2-1/2} \frac{dt}{t}
\end{equation}
Now from equation (17),
\[
a_n = (m_1 \tau_{1,1})^2 + \cdots + \left( \sum_{i=1,\ldots,n-2} m_i \tau_{i,n-2} \right)^2 + \left( \sum_{i=1,\ldots,n-2} m_i \tau_{i,n-1} \right) + m_{n-1} \tau_{n-1,n-1}
\]
\begin{equation}
= a_{n-1} + (b_{n-1} + c_{n-1} m_{n-1})^2.
\end{equation}
So using formula (20) again as above, we have
\[
\sum_{m_{n-1}} \exp(-\pi a_n t)
\]
\[
= \exp(-\pi t a_{n-1}) \sum_{m_{n-1}} \exp(-\pi t (b_{n-1} + c_{n-1} m_{n-1})^2)
\]
\[
= \frac{1}{c_{n-1} \sqrt{t}} \exp(-\pi t a_{n-1}) \sum_{m_{n-1}} \exp(2 \pi i b_{n-1} m_{n-1} / c_{n-1}) \exp(-\pi m_{n-1}^2 / t c_{n-1}^2).
\]
Putting this in (21),
\[
S_2 = \frac{1}{c_{n-1} c_n} \sum_{m_1 \neq 0, m_n} \exp(2 \pi i b_n m_n / c_n) \int_0^\infty \exp(-\pi m_{n-1}^2 / t) \sum_{m_{n-2}} \frac{1}{\sqrt{t}} \exp(-\pi t a_{n-1})
\]
\[
\cdot \sum_{m_{n-1}} \exp(2 \pi i (b_n m_n / c_n + b_{n-1} m_{n-1} / c_{n-1}))
\]
\[
\cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2 + m_{n-1}^2 / c_{n-1}^2) / t) \sum_{m_{n-2}} \exp(-\pi t a_{n-1}) t^{n_{s-2} / 2} dt
\]
\[
= \frac{1}{c_{n-1} c_n} \sum_{m_1 \neq 0, m_n} \exp(2 \pi i (b_n m_n / c_n + b_{n-1} m_{n-1} / c_{n-1}))
\]
\[
\cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2 + m_{n-1}^2 / c_{n-1}^2) / t) \sum_{m_{n-2}} \sum_{m_{n-3}} \exp(-\pi t a_{n-1}) t^{n_{s-2} / 2} dt
\]
(22)

Looking at equations (21) and (22), we see that repeated use of Poisson summation gives
\[
S_2 = \frac{1}{\prod_{i=2}^{n} c_i} \sum_{m_1 \neq 0, m_n, m_{n-1}, \ldots, m_2} \exp(2 \pi i d) \int_0^\infty \exp(-\pi m^2 / t) \exp(-\pi t a_1) t^{n_{s-2} / 2 - (n-1) / 2} dt,
\]
where recall that \(m\) and \(d\) were defined in Theorem 1.1. Now \(a_1 = m_2^2 y_2^2\) where \(y = \tau_1 = y_1 y_2 \cdots y_{n-1}\). So
\[
\left(\prod_{i=2}^{n} c_i\right) S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \ldots, m_n} \exp(2 \pi i d) \int_0^\infty \exp(-\pi (m_1 y)^2 t + \pi m^2 / t) t^{n_{s-1} / 2 + 1 / 2} dt.
\]

Denote the term corresponding \(m_2 = \cdots = m_n = 0\) by \(S'_2\), i.e.,
\[
S'_2 = \sum_{m_1 \neq 0} \int_0^\infty \exp(-\pi (m_1 y)^2 t) t^{n_{s-1} / 2 + 1 / 2} dt.
\]

Using formula (15), with \(s\) replaced by \(n(s - 1) / 2 + 1 / 2\), we get
\[
S'_2 = \sum_{m_1 \neq 0} \frac{\pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2)}{(m_1 y) 2^{(n(s-1)/2+1/2)}}
\]
\[
= y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2) \cdot 2 \sum_{m_1 \geq 0} \frac{1}{m_{n(s-1)+1}}
\]
\[
= 2y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2) \zeta(n(s-1) + 1)
\]
(23)
Let
\[ S''_2 = \left( \prod_{i=2}^{n} c_i \right) S_2 - S'_2 \]
\[ = \sum_{m_1 \neq 0} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp(2\pi id) \int_{0}^{\infty} \exp\left(-\left(\pi(m_1 y)^2 + \pi m^2/t\right)\right)^n t^{(s-1)/2+1/2} dt. \]  

For \( a \) and \( b \) positive real numbers, recall the function
\[ K_s(a, b) = \int_{0}^{\infty} \exp\left(-\left(a^2 t + b^2/\pi\right)\right) dt. \]

Noting that \( m \neq 0 \) if not all \( m_2, \ldots, m_n \) are zero,
\[ S''_2 = \sum_{m_1 \neq 0} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp(2\pi id) K_{n(s-1)/2+1/2}(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|). \]  

From equations (13), (14), (23), (24), and (25), we finally get an expression for \( E^*_{n}(\tau, s) \):
\[ E^*_{n}(\tau, s) = \left( \prod_{i=1}^{n-1} y_i^{(n-1)_i} \right)^s \cdot E^*_{n-1}\left( \tau', \frac{n}{n-1}s \right) + 2 \left( \prod_{i=1}^{n-1} y_i^{n-1}_i \right)^s \frac{1}{\prod_{i=2}^{n} c_i} y^{(n(s-1)+1)} \sum_{m_1 \neq 0} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp(2\pi id) K_{n(s-1)/2+1/2}(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|). \]  

The good thing about the formula above is that it is easy to read off the polar part and the constant term in each of the summands above, which is what we do now. It is known that \( K_s \) is an entire function of \( s \), and so all the functions appearing in the expression above are holomorphic at \( s = 1 \) except \( \zeta(2s-1) \), which has a simple pole at \( s = 1 \), and perhaps \( E^*_{n-1}(\tau', \frac{n}{n-1}s) \). By induction, \( E^*_{n-1}(\tau', \frac{n}{n-1}s) \) is also holomorphic except perhaps when \( \frac{n}{n-1}s = 1 \), and in particular is homomorphic at \( s = 1 \). So the first and last summands on the right side of equation (26) are holomorphic at \( s = 1 \); using the fact that \( K_{1/2}(a, b) = \frac{\sqrt{\pi}}{a} \exp(-2ab) \), their sum is
\[ \left( \prod_{i=1}^{n-1} y_i^{(n-1)_i} \right) E^*_{n-1}(\tau', \frac{n}{n-1}s) + \sum_{m_1 \neq 0} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp(2\pi id) \frac{1}{|m_1|} \exp(-2\pi|m_1||m|y) + O(s-1), \]

which is \(-4 \log g(\tau) + O(s-1)\).

In order to deal with the second summand on the right side of equation (26), note that
\[ \zeta(n(s-1) + 1) = \frac{1}{n(s-1)} + \gamma + O(s-1), \]
\[ \Gamma(n(s-1)/2 + 1/2) = \sqrt{\pi}(1 + \frac{n}{2}(\gamma - \log 4)(s-1) + O(s-1)^2), \]
\[ \pi^{-(n(s-1)/2+1/2)} = \frac{1}{\sqrt{\pi}} (1 - \frac{n}{2} \log \pi (s - 1) + O(s - 1)^2), \]
\[ y^{-(n(s-1)+1)} = y^{-1}(1 - n \log y(s - 1) + O(s - 1)^2), \]

and
\[ \left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^S = \left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^{(s-1)+1} = \left( \prod_{i=1}^{n-1} y_i^{n-i} \right) (1 + \log \left( \prod_{i=1}^{n-1} y_i^{n-i} \right) (s - 1) + O(s - 1)^2). \]

Using the formulas above, the second summand in on the right side of equation (26) becomes
\[ \frac{2}{n} \frac{y}{s-1} + \left( \frac{\gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i^2 \right)}{s-1} \right) + O(s-1) \]

Using the formulas obtained above for the three summands on the right side of equation (26), we get the formula for \( E_n^s(\tau, s) \) given in Theorem 1.1.

References


