A generalization of Kronecker’s first limit formula to GL(n)

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For $\tau \in \mathbb{H}$, the upper half plane, let $y = \Im(\tau)$.

Define $E^*_{\text{2}}(\tau, s) = \pi - s \Gamma(s) \sum_{m_1, m_2} y s |m_1 \tau + m_2|^2$.

It converges when $\Re(s) > 1$ and is known to have a meromorphic continuation to all $s \in \mathbb{C}$, with only pole at $s = 1$.

The classical Kronecker’s first limit formula says

$E^*_{\text{2}}(\tau, s) = \frac{1}{s - 1} + (\gamma - \log 4\pi - \log y - 4 \log |\eta(\tau)|) + O(s - 1),$

where $\gamma$ is the Euler-Mascheroni constant and $\eta(z)$ is the Dedekind eta-function.

Kronecker’s first limit formula has several applications; we mention one such next.
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For $\tau \in \mathcal{H}$, the upper half plane, let $y = \Im(\tau)$, and for $s \in \mathbb{C}$, define $E^*_2(\tau, s) = \pi^{-s}\Gamma(s)\sum_{m_1, m_2} \frac{y^s}{|m_1\tau + m_2|^{2s}}$;
For $\tau \in \mathcal{H}$, the upper half plane, let $y = \Im(\tau)$, and for $s \in \mathbb{C}$, define $E_2^*(\tau, s) = \pi^{-s} \Gamma(s) \sum_{m_1, m_2} \frac{y^s}{|m_1 \tau + m_2|^{2s}}$; it converges when $\Re(s) > 1$.
Classical Kronecker’s first limit formula

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The classical Kronecker’s first limit formula says $E_2^*(\tau, s) = \frac{1}{s-1} + \frac{(\gamma - \log 4\pi - \log y - 4 \log |\eta(\tau)|)}{s} + O(s-1)$, where $\gamma$ is the Euler-Mascheroni constant and $\eta(z)$ is the Dedekind eta-function.

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$$
\zeta_K(s) = \frac{1}{w_K 2\pi \sqrt{d_K}} \left( \frac{1}{s-1} + 2\gamma - \log 2 - \log y - 4 \log |\eta(\tau)| \right) + O(|s-1|),
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where $\tau$ is an element of the upper half plane such that $\{1, \tau\}$ is a basis for an ideal in the inverse class of $A$, and $y$ is the imaginary part of $\tau$. 


Let $K$ be a number field. If $A$ is an ideal class of $K$, then the partial zeta function associated to $A$ is $\zeta_K(s, A) = \sum_{a\in A} \frac{1}{N_a s}$, and the Dedekind zeta function is $\zeta_K(s) = \sum_a \frac{1}{N_a s} = \sum_A \zeta_K(s, A)$; these series converge when $\text{Re}(s) > 1$ and are known to have meromorphic continuations to all $s \in \mathbb{C}$, with only pole at $s = 1$.

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Dedekind zeta functions of imaginary quadratic fields

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Using the first limit formula, Kronecker showed that

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where $\tau$ is an element of the upper half plane.
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Now let $K$ be a real quadratic field.
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Using Kronecker’s first limit formula, Hecke showed that
\[
\frac{1}{2} (\pi^{-1} d_K^{1/2})^s \Gamma(s/2)^2 \zeta_K(s, A) =
\]
\[
\log \epsilon^s - 1 + (\gamma - \log 4 \pi) \log \epsilon - \int \log y(t) \frac{dt}{t} - 4 \int \log |\eta(\tau(t))| \frac{dt}{t} + O(s^{-1}),
\]
where \( \epsilon \) is a fundamental unit of \( K \),
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\( \epsilon \) is a fundamental unit of \( K \),

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\eta(\tau(t)) = \frac{\tau(t)}{\tau(1)}.
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Generalization of Kronecker’s first limit formula

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Consider for $\tau \in \mathcal{H}^n$ and $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$E_n(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} \frac{(\det \tau)^s}{\|(m_1 \ldots m_n)\tau\|^s/2}.$$
For \( n \geq 2 \), the generalized upper half-plane \( \mathcal{H}^n = \text{GL}_n(\mathbb{R})/O_n(\mathbb{R})\mathbb{R}^\times \); consists of certain \( n \times n \) matrices; note that \( \mathcal{H}^2 \) is the usual upper half plane, and the point \( x + iy \) corresponds to the matrix \[
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Consider for \( \tau \in \mathcal{H}^n \) and \( s \in \mathbb{C} \) with \( \Re(s) > 1 \),
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E_n^*(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} \frac{\left| (\det \tau)^s \right|}{\| (m_1 \ldots m_n) \tau \|^{|ns/2|}}.
\]

**Theorem (Liu-Masri, A)**

\( E_n^*(\tau, s) \) has meromorphic continuation to all \( s \in \mathbb{C} \) with the only pole at \( s = 1 \),
For $n \geq 2$, the generalized upper half-plane $\mathcal{H}^n = GL_n(\mathbb{R})/O_n(\mathbb{R})\mathbb{R}^\times$; consists of certain $n \times n$ matrices; note that $\mathcal{H}^2$ is the usual upper half plane, and the point $x + iy$ corresponds to the matrix

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Consider for $\tau \in \mathcal{H}^n$ and $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$E_n^*(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} (\det \tau)^s \|m_1 \cdots m_n\tau\|^s/n.$$ 

**Theorem (Liu-Masri, A)**

$E_n^*(\tau, s)$ has meromorphic continuation to all $s \in \mathbb{C}$ with the only pole at $s = 1$, and

$$E_n^*(\tau, s) = \frac{2/n}{s-1} + \left( \gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i \right) - 4 \log g(\tau) \right) + O(s-1),$$

where

$y_i$'s are related to the diagonal entries mentioned above in $\tau$, and $g(\tau)$ is an explicit function (it generalizes $|\eta(\tau)|$). Our proof is self contained; the proof of Liu-Masri relies on work of Terras (all use the Poisson summation formula).
Generalization of Kronecker’s first limit formula

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Consider for $\tau \in \mathfrak{H}^n$ and $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$E^*_n(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} (\det \tau)^s ||(m_1 \ldots m_n)\tau||^{ns/2} .$$

**Theorem (Liu-Masri, A)**

$E^*_n(\tau, s)$ has meromorphic continuation to all $s \in \mathbb{C}$ with the only pole at $s = 1$, and

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Consider for \( \tau \in \mathcal{H}^n \) and \( s \in \mathbb{C} \) with \( \Re(s) > 1 \),

\[
E^*_n(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n) \tau \|^{ns/2}}.
\]

Theorem (Liu-Masri, A)

\( E^*_n(\tau, s) \) has meromorphic continuation to all \( s \in \mathbb{C} \) with the only pole at \( s = 1 \), and

\[
E^*_n(\tau, s) = \frac{2/n}{s-1} + \left( \gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i \right) - 4 \log g(\tau) \right) + O(s-1),
\]

where \( y_i \)'s are related to the diagonal entries mentioned above in \( \tau \),

and \( g(\tau) \) is an explicit function (it generalizes \( |\eta(\tau)| \)).
Generalization of Kronecker’s first limit formula

For $n \geq 2$, the generalized upper half-plane $\mathcal{H}^n = \text{GL}_n(\mathbb{R})/O_n(\mathbb{R})\mathbb{R}^\times$; consists of certain $n \times n$ matrices; note that $\mathcal{H}^2$ is the usual upper half plane, and the point $x + iy$ corresponds to the matrix $\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$

Consider for $\tau \in \mathcal{H}^n$ and $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$E_n^*(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} (\det \tau)^s \frac{|(m_1 \ldots m_n)\tau|^ns/2}{\| (m_1 \ldots m_n)\tau \|^ns/2}.$$ 

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$E_n^*(\tau, s)$ has meromorphic continuation to all $s \in \mathbb{C}$ with the only pole at $s = 1$, and

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Our proof is self contained; the proof of Liu-Masri relies on work of Terras (all use the Poisson summation formula).
One can define

\[ \eta(\tau) = q_1 \prod_{(\ell_2, \ldots, \ell_n)} (1 - q_2) \exp(2\pi iv), \]

where \( q_1, q_2, v \) are certain explicit functions of \( \tau \) and \( (\ell_2, \ldots, \ell_n) \), so that \( g(\tau) = |\eta(\tau)| \) and \( \eta(\tau) \) coincides with the usual Dedekind eta function when \( \ell = 2 \).

The classical Kronecker’s first limit formula can be used to show the automorphy property of the usual Dedekind eta function, and probably our generalization of Kronecker’s first limit formula can be used to show the automorphy property of our generalization of the Dedekind eta function; however: are there any applications of the generalization of the Dedekind eta function?
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Let $K$ be a number field of degree $n > 2$. 
Dedekind zeta functions of arbitrary number fields

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**Theorem**

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\]

\[
\tau: \mathbb{R}^m \to \mathbb{H}^n
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is an explicit function, $r$ denotes the number of real embeddings of $K$, $c$ denotes the number of complex conjugate embeddings, $m = r + c - 1$, $\epsilon_1, \ldots, \epsilon_m$ denotes a fundamental set of units of $K$, $D$ is a fundamental domain under the action of $\langle \epsilon_1, \ldots, \epsilon_m \rangle$ on $(\mathbb{R}^n_{>0})^m$, $V = \int_D dt_1 \cdots dt_m$, and $d(s)$ is an explicit function.

Was proved for totally real fields by Liu-Masri. Both proofs use generalizations of a trick of Hecke (was done for cubic fields by Efrat).
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Thank you!