A generalization of Kronecker's first limit formula to GL(n)

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Our proof is self contained;

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For $n \geq 2$, the generalized upper half-plane $\mathfrak{H}^n = GL_n(\mathbf{R})/O_n(\mathbf{R})\mathbf{R}^\times$; consists of certain $n \times n$ matrices; note that $\mathfrak{H}^2$ is the usual upper half plane, and the point $x + iy$ corresponds to the matrix $\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$

Consider for $\tau \in \mathfrak{H}^n$ and $s \in \mathbf{C}$ with $\Re(s) > 1$,
$E_n^*(\tau, s) = \pi^{-ns/2}\Gamma(ns/2)\sum_{m_1,\ldots,m_n} \frac{(\det \tau)^s}{\|(m_1 \ldots m_n)\tau\|^{ns/2}}$.

Theorem (Liu-Masri, A)

$E_n^*(\tau, s)$ has meromorphic continuation to all $s \in \mathbf{C}$ with the only pole at $s = 1$, and

$$E_n^*(\tau, s) = \frac{2/n}{s-1} + \left(\gamma - \log 4\pi - \frac{2}{n}\log\left(\prod_{i=1}^{n-1} y_i^i\right) - 4\log g(\tau)\right) + O(s-1) ,$$

where y_i's are related to the diagonal entries mentioned above in τ, and $g(\tau)$ is an explicit function (it generalizes $|\eta(\tau)|$).
Our proof is self contained; the proof of Liu-Masri relies on work of Terras (all use the Poisson summation formula).

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however: are there any applications of the generalization of the Dedekind eta function??

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Was proved for totally real fields by Liu-Masri. Both proofs use generalizations of a trick of Hecke (was done for cubic fields by Efrat).

Thank you!