A generalization of Kronecker’s first limit formula with application to zeta functions of number fields

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Classical Kronecker’s first limit formula

Let $H = \{ \tau \in \mathbb{C} : \text{Im}(\tau) > 0 \}$. For $\tau \in H$ and $s \in \mathbb{C}$, define the Eisenstein series $E^*_{2}(\tau, s) = \pi^{-s} \Gamma(s) \sum_{m, n} \frac{\text{Im}(\tau)}{|m\tau + n|^2} s$; it converges when $\text{Re}(s) > 1$ and is known to have a meromorphic continuation to all $s \in \mathbb{C}$, with only pole at $s = 1$.

The classical Kronecker’s first limit formula gives the Laurent series expansion near $s = 1$:

$$E^*_{2}(\tau, s) = \frac{1}{s - 1} + \left( \gamma - \log 4\pi - \log \text{Im}(\tau) - 4 \log |\eta(\tau)| \right) + O(s - 1),$$

where $\gamma$ is the Euler-Mascheroni constant and $\eta(z)$ is the Dedekind eta-function.

One can deduce the modularity property of $\eta(z)$ using the formula above.

Kronecker’s first limit formula has many other applications; we mention one such next.
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E_2^*(\tau, s) = \pi^{-s} \Gamma(s) \sum_{m,n} \frac{\text{Im}(\tau)^s}{|m\tau+n|^2^s};
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it converges when \( \text{Re}(s) > 1 \) and is known to have a meromorphic continuation to all \( s \in \mathbb{C} \), with only pole at \( s = 1 \).

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Let $K$ be a number field. The Dedekind zeta function of $K$ is 
\[ \zeta_K(s) = \sum_a \frac{1}{N\alpha^s}, \]
where $a$ runs over the ideals of $K$. 

If $A$ is an ideal class of $K$, then the partial zeta function associated to $A$ is 
\[ \zeta_K(s, A) = \sum_{a \in A} \frac{1}{N\alpha^s}, \]
so that 
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These series converge when $\text{Re}(s) > 1$ and are known to have meromorphic continuations to all $s \in \mathbb{C}$, with only pole at $s = 1$. 

What are the Laurent series expansions of these functions near $s = 1$? 

Now let $K$ be an imaginary quadratic field. Using the first limit formula, Kronecker showed that 
\[ \zeta_K(s, A) = \frac{1}{s-1} + \left(2\gamma - \log 2 - \log \text{Im}(\tau) - 4\log |\eta(\tau)| + O(s-1) \right), \]
where $w_K$ denotes the number of roots of unity in $K$, $d_K$ denotes the discriminant of $K$, $\tau$ is an element of the upper half plane such that \{1, $\tau$\} is a basis for an ideal in the inverse class of $A$, and $y$ is the imaginary part of $\tau$. 

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Using Kronecker's first limit formula, Hecke showed that

$$\frac{1}{2} \left( \pi^{-1} d_K^{1/2} \right)^s \Gamma(s/2)^2 \zeta_K(s, A) =$$

$$\frac{\log \epsilon}{s-1} + \left( (\gamma - \log 4\pi) \log \epsilon - \int_1^\epsilon \log y(t) \frac{dt}{t} - 4 \int_1^\epsilon \log |\eta(\tau(t))| \frac{dt}{t} \right) + O(s - 1),$$

where $\epsilon$ is a fundamental unit of $K$, $\tau(t)$ is a certain curve in the upper half plane (depends on $A$), and $y(t)$ denotes its $y$-coordinate.

The preceding formula was generalized by Bump-Goldfeld to real cubic fields, Efrat to all cubic fields, and Liu-Masri to all totally real fields. We generalize it to all number fields.
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Consider for $\tau \in \mathcal{H}^n$ and $s \in \mathbb{C}$ with $\Re(s) > 1$,

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Generalization of Kronecker’s first limit formula

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**Theorem (Liu-Masri, A)**

$E_n^*(\tau, s)$ has meromorphic continuation to all $s \in \mathbb{C}$ with the only pole at $s = 1$, and

$$E_n^*(\tau, s) = \frac{2}{n^s} - \left(\gamma - \log 4\pi - 2\pi \log \prod_{i=1}^{n-1} y_i\right) + O(s - 1),$$

where $y_i$'s are related to the diagonal entries in $\tau$, and $g(\tau)$ is an explicit function (it generalizes $|\eta(\tau)|$).

Our proof is self contained; the proof of Liu-Masri relies on work of Terras (all use the Poisson summation formula).
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$E^*_n(\tau, s)$ has meromorphic continuation to all $s \in \mathbb{C}$ with the only pole at $s = 1$, and

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**Theorem**

$$w_K \left(2^{-c} \pi^{-n/2} d_K^{1/2}\right)^s d(s) \Gamma(ns/2) \zeta_K(s, A) =$$

$$\frac{2V/n}{s-1} + (\gamma - \log 4\pi)V - \frac{2}{n} \int_D \log \left(\prod_{i=1}^{n-1} y_i(t_1, \ldots, t_m)^i\right) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} -$$

$$4 \int_D \log g(\tau(t_1, \ldots, t_m)) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} + O(s - 1),$$

where $\tau$ is an explicit function, $r$ denotes the number of real embeddings of $K$, $c$ denotes the number of complex conjugate embeddings, $m = r + c - 1$, $\epsilon_1, \ldots, \epsilon_m$ denotes a fundamental set of units of $K$, $D$ is a fundamental domain under the action of $\langle \epsilon_1, \ldots, \epsilon_m \rangle$ on $(\mathbb{R}^0)^m$, $V = \int_D dt_1 \cdots dt_m$, and $d(s)$ is an explicit function.

Was proved for cubic fields by Efrat and for totally real fields by Liu-Masri. All proofs use generalizations of a trick of Hecke.
Dedekind zeta functions of arbitrary number fields

Let $K$ be a number field of degree $n > 2$.

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where $\tau : \mathbb{R}^m \to \mathfrak{H}^n$ is an explicit function, $r$ denotes the number of real embeddings of $K$, $c$ denotes the number of complex conjugate embeddings, $m = r + c - 1$, $\epsilon_1, \ldots, \epsilon_m$ denotes a fundamental set of units of $K$, $D$ is a fundamental domain under the action of $\langle \epsilon_1, \ldots, \epsilon_m \rangle$ on $(\mathbb{R}_{>0})^m$, $V = \int_D \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}$, and $d(s)$ is an explicit function.
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\[
\begin{align*}
&\frac{w_K}{2} \left(2^{c \pi} - n/2 d_K^{1/2}\right)^s d(s) \Gamma(ns/2) \zeta_K(s, A) = \\
&\frac{2V}{n(s-1)} + (\gamma - \log 4\pi)V - \frac{2}{n} \int_D \log \left(\prod_{i=1}^{n-1} y_i(t_1, \ldots, t_m)^i\right) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} - \\
&4 \int_D \log g(\tau(t_1, \ldots, t_m)) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} + O(s - 1),
\end{align*}
\]

where $\tau: \mathbb{R}^m \to \mathfrak{H}^n$ is an explicit function, $r$ denotes the number of real embeddings of $K$, $c$ denotes the number of complex conjugate embeddings, $m = r + c - 1$, $\epsilon_1, \ldots, \epsilon_m$ denotes a fundamental set of units of $K$, $D$ is a fundamental domain under the action of $\langle \epsilon_1, \ldots, \epsilon_m \rangle$ on $(\mathbb{R}_{>0})^m$, $V = \int_D \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}$, and $d(s)$ is an explicit function.

Was proved for cubic fields by Efrat and for totally real fields by Liu-Masri. All proofs use generalizations of a trick of Hecke.
Thank you!