Visibility of Shafarevich-Tate groups *

Amod Agashe
University of Texas, Austin

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*These slides can be obtained from
http://www.math.utexas.edu/~amod/mymath.html

Definition of visibility

$$\begin{array}{rcl} K & = & \text{a number field} \\ E & = & \text{an elliptic curve over } K \\ Ш(E) & = & \text{Shafarevich-Tate group of } E \\ & = & \text{isomorphism classes of torsors for } E \\ & & \text{that are locally trivial everywhere} \end{array}$$

B. Mazur: how can one "visualize" the curves of genus 1 that represent elements of $Ш(E)$?

Suppose we are given an embedding over K of E into an abelian variety J.

Definition 1 (Mazur, at AWS 98).
An element σ of $Ш(E/K)$ is said to be visible *in J if it is in the kernel of the natural homomorphism $Ш(E/K) \to Ш(J/K)$.*

Under certain conditions, this definition is equivalent to the statement that the curve of genus 1 that represents σ is isomorphic over K to a curve contained in the variety J, hence the terminology "visible".

The definition generalizes easily to abelian varieties of arbitrary dimension.

Shimura quotients

$$\begin{array}{rcl} N & = & \text{a positive integer} \\ J_0(N) & = & \text{Jacobian of the modular curve } X_0(N) \\ \mathbf{T} & = & \text{Hecke algebra} \\ f & = & \text{a newform} \\ I_f & = & \text{Ann}_{\mathbf{T}} f\text{, an ideal of } \mathbf{T} \\ A = A_f & = & J_0(N)/I_f J_0(N), \\ & & \text{the Shimura quotient associated to } f \end{array}$$

Then $A^\vee$ is an abelian subvariety of $J_0(N)$ and we can ask about the visibility of elements of $Ш(A^\vee)$ in $J_0(N)$. We will mainly be concerned with such abelian varieties for the rest of this talk.

How can we decide if an element of $Ш(A^\vee)$ is visible in $J_0(N)$ or not?

Detecting existence of invisible elements

Consider the composite $A^{\vee} \hookrightarrow J_0(N) \twoheadrightarrow A$, which is an isogeny; call it ϕ_A.

Lemma 2 (Mazur, AA).
Every visible element of $Ш(A^{\vee})$ *is killed by multiplication by the exponent of* $\ker\phi_A$.

Assume, as conjectured, that Ш is finite. If a prime divides the order of $Ш(A)$ but doesn't divide $\deg(\phi_A)$, then $Ш(A)$ has invisible elements.

For computing the structure of $\ker\phi_A$:
Cremona for elliptic curves, Stein for higher dimensions.

For computing the order of $Ш(A)$, assume and use the Birch and Swinnerton-Dyer formula for $L_A^{(r)}(1)/\Omega(A)$.
For elliptic curves: Birch, Manin; Cremona.
For higher dimensions, when $L_A(1) \neq 0$:
AA, Stein.

Proving existence of visible elements

Theorem 3 (Mazur).
If E is an elliptic curve over a number field K and σ is an element of $Ш(E)$ of order 3 , then there is an abelian surface J over K such that σ is visible in J.

Theorem 4 (de Jong, Stein). *If A is an abelian variety over a number field K and σ is an element of $Ш(A)$, then there exists an abelian variety J over K containing A as an abelian subvariety such that σ is visible in J.*

Theorem 5 (Mazur, Stein).
Let C be an abelian subvariety of $J_0(N)$ of rank 0.
Suppose D is another abelian subvariety of $J_0(N)$ and p is a prime s.t. $D[p] \subseteq C$.
(Call D a p-complementary abelian variety to C.)
Then, under certain additional hypotheses, there is an injection of $D(\mathbf{Q})/pD(\mathbf{Q})$ into the visible part of $Ш(C)$. (So $\mathrm{rk}(D(\mathbf{Q})) > 0 \Rightarrow$ non-trivial visible elements of $Ш(C)$.)

Prime levels with $\#Ш_{\mathrm{an}}(A_f) > 1$

(Calculated by William Stein)
Warning: only odd parts of the invariants are shown.

$\mathbf{A_f}$	$\#Ш_{\mathrm{an}}(A_f)$	$\sqrt{deg(\phi_{A_f})}$	$\mathbf{B_g}$
389E	5^2	5	**389A**
433D	7^2	$3 \cdot 7 \cdot 37$	**433A**
563E	13^2	13	**563A**
571D	3^2	$3^2 \cdot 127$	**571B**
709C	11^2	11	**709A**
997H	3^4	3^2	**997B**
1061D	151^2	$61 \cdot 151 \cdot 179$	**1061B**
1091C	7^2	1	NONE
1171D	11^2	$3^4 \cdot 11$	**1171A**
1283C	5^2	$5 \cdot 41 \cdot 59$	none
...			
2111B	211^2	1	NONE
...			
2333C	83341^2	83341	**2333A**
...			

$\#Ш_{\mathrm{an}}(A_f)$ = order of the Shafarevich-Tate group as predicted by the BSD formula.

B_g = an optimal quotient of $J_0(N)$ with $L(B_g, 1) = 0$ s.t. if an odd prime p divides $\#Ш_{\mathrm{an}}(A_f)$, then $B_g^\vee[p] \subseteq A_f^\vee$.

Example 6 (Stein). $5^2 \mid \#Ш_{389E}$.

History

Warning: All results on invisibility are conjectural (BSD), and we consider only the odd part of Ш for simplicity.

Logan, Mazur: Elliptic curves of square-free conductor < 3000. Could detect invisibility only for $2849A$; all others have complementary elliptic curve.

AA, Merel: Winding quotients of prime level. Upto level 1400, detected invisibility only for level 1091; not visible in $J_1(1091)$ either.

Cremona, Mazur: Extended to elliptic curves of conductors < 5500. Detected invisibility only in 3 out of 52 cases, found complementary elliptic curve in 43 cases. (Stoll: need not assume BSD)

Stein: Shimura quotients of prime level < 5650 and non-zero special L-value.
Invisible 90% of the time between 2600 & 5650.

Future

Mazur, Merel: For every element σ of $Ш(A_f)$, is there an M s.t. σ is visible in $J_0(NM)$?
E.g., Stein: $Ш(2849A)$ is visible in $J_0(3{\cdot}2849)$.

Eventual visibility (Merel, Stein): Is the direct limit of $Ш(J_0(N))$ trivial?

Stein: Visibility of elements of Mordell-Weil groups.
Use complementary abelian varieties to prove non-triviality of ranks of abelian varieties, by proving non-triviality of $Ш$ using Euler systems (Kolyvagin, McCallum).