On the notion of visibility of torsors

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Abstract

Let J be an abelian variety and A be an abelian subvariety of J, both defined over $\mathbf{Q}$. Let x be an element of $H^1(\mathbf{Q}, A)$. Then there are at least two definitions of x being visible in J: one asks that the torsor corresponding to x is isomorphic over $\mathbf{Q}$ to a subvariety of J, the other asks that x is in the kernel of the natural map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. In this article, we clarify the relation between the two definitions.

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1 Introduction and definitions

As in the abstract, let J be an abelian variety and A be an abelian subvariety of J, both defined over $\mathbf{Q}$. The concept of visibility of torsors of A (i.e., elements of $H^1(\mathbf{Q}, A)$) was introduced by Mazur [Maz98] in the context where J is the Jacobian of a modular curve and A is an elliptic curve. He was interested in visualizing elements of the Shafarevich-Tate group of A, which is a subgroup of $H^1(\mathbf{Q}, A)$, as subvarieties in an ambient space (i.e., describing them geometrically, as opposed to just algebraically). Apart from $\mathbf{P}^n$ for some n, the other natural choice for the ambient space was the abelian variety J, where A is already a subvariety. The theory that the notion of visibility led to has provided much computational and theoretical evidence for the second part of the Birch and Swinnerton-Dyer conjecture (see [CM00, AS05, Aga10b, Aga09b, Aga10a, Aga09a]).

Following Mazur's original motivation, we make the following definition:

Definition 1.1. *An element of of $H^1(\mathbf{Q}, A)$ is said to be visible as a variety in J if it is isomorphic over $\mathbf{Q}$ to a subvariety of J.*

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In the applications of the notion of visibility to the Birch and Swinnerton-Dyer conjecture (e.g., [AS05]), another definition of visibility has been used, which has become the standard definition, and is as follows:

Definition 1.2. *We say that an element of $H^1(\mathbf{Q}, A)$ is visible in J if it is in the kernel of the map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$ induced by the inclusion of A in J.*

Note that Definition 1.2 is algebraic in nature, while Definition 1.1 is geometric. The first goal of this article is to relate these two definitions, and thus give a geometric interpretation of visible elements (which also explains the use of the word "visible" in Definition 1.2 above). In order to do so, we introduce yet another notion of visibility as follows:

Definition 1.3. *Let x be an element of $H^1(\mathbf{Q}, A)$ and let V denote the corresponding torsor. We say that x (or V) is visible as a torsor in J if there is a subvariety V' of J such that the group law of J induces an action of A on V', and there is an isomorphism of A-torsors $\iota : V \xrightarrow{\cong} V'$ (i.e., an isomorphism of varieties over $\mathbf{Q}$ that respects the action of A).*

We show in Proposition 2.1 below that an element of $H^1(\mathbf{Q}, A)$ is visible in J if and only if it is visible as a torsor. It is clear that if an element of $H^1(\mathbf{Q}, A)$ is visible as a torsor in J then it is visible as a variety in J; in particular, if it is visible, then it is visible as a variety. We do not know if the converse is true in general; however we do give some conditions under which the converse holds – see Proposition 3.1 below.

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2 Visibility as a torsor

The goal of this section is a proof of the following proposition:

Proposition 2.1. *Recall that J is an abelian variety and A is an abelian subvariety of J, both defined over $\mathbf{Q}$. Let V be an A-torsor. Then V is visible as a torsor in J if and only if it is visible in J (i.e., the cocycle class corresponding to V is in the kernel of the natural map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$).*

It is convenient to use the notion of sheaf torsors (see [Mil80, § III.4]). If A is an abelian variety over $\mathbf{Q}$, let ST(A) denote the equivalence classes of sheaf torsors of A. If V is a sheaf torsor, pick $P \in V(\overline{\mathbf{Q}})$. Corresponding to P, we have a cocycle given by $\sigma \mapsto \sigma(P) - P \in A(\overline{\mathbf{Q}})$ for $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. One can show that this gives an element of $H^1(\mathbf{Q}, A)$ that is independent of the choice of the point P above. Thus we get a canonical map $\mathrm{ST}(A) \to H^1(\mathbf{Q}, A)$. By Theorems 1.7, 3.9. 2.10, and 4.6 in Chapter III of [Mil80], this map is an isomorphism.

In this section, the letter R will stand for a $\mathbf{Q}$-algebra of finite type. If V is an A-sheaf torsor, then recall that the *pushout* $V \times^A J$ is the sheaf whose sections over R are the set of orbits of $V(R) \times J(R)$ under the action of $A(R)$, where $A(R)$ acts on $V(R)$ in the usual way, but on $J(R)$ the action is by the inverse of the group law on $J(R)$. Also $V(R) \times J(R)$ has an action of $J(R)$ on the second component, which is compatible with the $A(R)$ action. Thus we have an action of $J(R)$ on $(V \times^A J)(R)$, and so $V \times^A J$ is a J-torsor.

The map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$ induces a map $\mathrm{ST}(A) \to \mathrm{ST}(J)$. We first claim that the image of the sheaf torsor corresponding to V under this induced map is the pushout $V \times^A J$.

Proof of the claim. Pick $P \in V(\overline{\mathbf{Q}})$ and let $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Just for the proof of this claim, we shall write the torsor action as a function, i.e., if $a \in A(\overline{\mathbf{Q}})$ and $x \in V(\overline{\mathbf{Q}})$, then $a(x)$ stands for the image of a acting on x under the action of A on V. The cocycle in $H^1(\mathbf{Q}, A)$ corresponding to V maps σ to a_σ, where a_σ is the unique element of $A(\overline{\mathbf{Q}})$ such that $\sigma(P) = a_\sigma(P)$. Now consider the point $(P, 0) \in V(\overline{\mathbf{Q}}) \times J(\overline{\mathbf{Q}})$, and let Q be its image in $(V \times^A J)(\overline{\mathbf{Q}})$. Then an easy check shows that $\sigma(Q) = a_\sigma(Q)$, where now a_σ is considered an element of $J(\overline{\mathbf{Q}})$. So the cocycle in $H^1(\mathbf{Q}, J)$ corresponding to $V \times^A J$ maps σ to $a_\sigma \in J(\overline{\mathbf{Q}})$. But this is exactly the image of V under the map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. This proves the claim. □

Proof of Proposition 2.1. Suppose V is visible as a torsor in J, and let i denote the composite map $V \xrightarrow{\iota} V' \hookrightarrow J$, where ι and V' are as given by Definition 1.3. Then consider the map of sheaf torsors $j : V \to V \times^A J$ induced by the map on sections $V(R) \to V(R) \times J(R)$ given by $v \mapsto (v, -i(v))$. Let v_1 and v_2 be elements of $V(R)$. Then v_1 and v_2 differ by translation by an element of $A(R)$, and so $-i(v_1)$ and $-i(v_2)$ differ by translation by the same element of $A(R)$. Hence the images of v_1 and v_2 under the map j are the same. Thus the image of the map $V(R) \to (V \times^A J)(R)$ is a point. This point is also invariant under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ (since the map j is defined over $\mathbf{Q}$). Hence this gives us a point of $V \times^A J$ over $\mathbf{Q}$. But that makes $V \times^A J$ the trivial torsor. Hence by the claim above, the cocycle class corresponding to V in $H^1(\mathbf{Q}, A)$ maps to the trivial element of $H^1(\mathbf{Q}, J)$, which proves the "only if" part of Proposition 2.1.

In the other direction, suppose the cocycle class corresponding to V is in the kernel of the map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. By the claim above, this means that there is an isomorphism $\phi : V \times^A J \xrightarrow{\cong} J$ over $\mathbf{Q}$. Recall that R denotes a $\mathbf{Q}$-algebra of finite type and consider the map $\psi : V(R) \to (V \times^A J)(R)$ induced by the map $V(R) \to V(R) \times J(R)$ given by $v \mapsto (v, 0)$. An easy check shows that the composite $V(R) \xrightarrow{\psi} (V \times^A J)(R) \xrightarrow{\phi} J(R)$ is an injection and respects the action of $A(R)$. By Yoneda's lemma, we have a monomorphism (i.e., a closed immersion) $V \to J$ that respects the action of A. This shows that V is visible as a torsor in J, and finishes the proof of Proposition 2.1. □

3 Visibility as a variety

This section is a generalization of some results from [Maz98].

Let J be an abelian variety and A be an abelian subvariety of J, both defined over $\mathbf{Q}$. Consider the following condition on the pair (J, A):

$(*)$ if $J \sim A \times B$ is an isogeny over $\overline{\mathbf{Q}}$, then no simple factor of A (over $\overline{\mathbf{Q}}$) is isogenous (over $\overline{\mathbf{Q}}$) to a simple factor (over $\overline{\mathbf{Q}}$) of B.

The following result was stated without proof in [Aga99] (as Lemma 3.1 in loc. cit.).

Proposition 3.1. *Let A be an abelian subvariety of J satisfying $(*)$. Let V be an A-torsor that is visible as a variety in J. Let i denote the embedding of A in J and consider the natural map $\tilde{i} : H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. Then there exists an automorphism ϕ of A (defined over $\mathbf{Q}$) such that $\tilde{i}(\tilde{\phi}(V))$ is trivial, where $\tilde{\phi}$ is the automorphism of $H^1(\mathbf{Q}, A)$ induced by ϕ.*

Thus if the condition (*) holds, then a torsor is visible *as a variety* if and only if it is visible "up to an automorphism of A". The condition (*) is satisfied for example if J is the Jacobian of the modular curve $X_0(N)$ for some positive integer N and A is the abelian subvariety of J associated to a newform on $\Gamma_0(N)$ (see, e.g., the proof of Lemma 3.1 in [ARS]). This situation is the most important one for the application of the notion of visibility so far. In [CM00], the same situation was considered, with the added restriction that A is a semistable elliptic curve; in this case, the only automorphisms of A are multiplication by ± 1, and so all definitions of visibility coincide (cf. [CM00, Remark 2]).

Proof of Proposition 3.1. Suppose V is an A-torsor visible as a variety in J and let V' be the subvariety of J isomorphic to V over $\mathbf{Q}$ given by Definition 1.1. Let $\iota : V \to V'$ denote the isomorphism between V and V' (over $\mathbf{Q}$). Since V is an A-torsor, we have an isomorphism $\psi : A \stackrel{\simeq}{\to} V$ over $\overline{\mathbf{Q}}$. Consider the composite map $A \stackrel{\psi}{\to} V \stackrel{\iota}{\to} V' \to J/A$, defined over $\overline{\mathbf{Q}}$. Up to translation, it is a homomorphism of abelian varieties. Its image has to be a point because otherwise that would violate $(*)$. Hence the image of $V' \to J/A$ is also a point. Thus V' is a translate of A (over $\overline{\mathbf{Q}}$) and hence has an action of A by translation. As a torsor in $H^1(\mathbf{Q}, A)$, it is given by $\sigma \mapsto \sigma(Q) - Q$ for any $Q \in V'(\overline{\mathbf{Q}})$, where the subtraction is the usual subtraction in J. But this is the zero element in $H^1(\mathbf{Q}, J)$ (under $\tilde{i}$) since $Q \in V'(\overline{\mathbf{Q}}) \subseteq J(\overline{\mathbf{Q}})$. Thus $\tilde{i}(V') = 0$.

Next, let $P \in V(\overline{\mathbf{Q}})$. Then the element of $H^1(\mathbf{Q}, A)$ corresponding to V is $\sigma \mapsto \sigma(P) -_V P$ where we will be using subscripts to distinguish different actions of A. Then the element of $H^1(\mathbf{Q}, A)$ corresponding to V' is given by $\sigma \mapsto \sigma(\iota(P)) -_{V'} \iota(P)$. Consider the map $\phi : A \to A$ given by $a \mapsto \iota(P +_V a) -_{V'} \iota(P)$. It is defined over $\mathbf{Q}$ and it is a homomorphism of abelian varieties since it takes the identity element of A to itself. It takes the torsor V to V' and thus $\tilde{i}(\tilde{\phi}(V)) = \tilde{i}(V')$. But as shown above, $\tilde{i}(V') = 0$, and so $\tilde{i}(\tilde{\phi}(V)) = 0$. Also, ϕ is an automorphism since it has an inverse given by $a \mapsto \iota^{-1}(\iota(P) +_{V'} a) -_V P$. This finishes the proof. ☐

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