

The zeros of the Riemann zeta function and its generalization to modular forms

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It is known that the Riemann zeta function has infinitely many zeros on the critical line, and they are all conjectured to be simple. See graph.

Completed Riemann zeta function

Let $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, which we call the *completed zeta function*. It is different from $\xi(s) = (1/2)s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$.

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Question: has anyone graphed the zero loci of the real and imaginary parts of $\Lambda(s)$?

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Hecke:

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Completed L-functions of modular forms

Put $r = s - 1$ to shift the critical line to the imaginary axis, and define $\tilde{\Lambda}_f(r) = \Lambda_f(r + 1)$. We want to study the zeros of $\tilde{\Lambda}_f(r)$ on the imaginary axis. Let $r = u + iv$.

$$\begin{aligned}\tilde{\Lambda}_f(u + iv) &= \int_1^\infty (e^{u \ln t} + e^{-u \ln t}) \cos(v \ln t) f(it/\sqrt{N}) dt \\ &\quad + i \int_1^\infty (e^{u \ln t} - e^{-u \ln t}) \sin(v \ln t) f(it/\sqrt{N}) dt\end{aligned}$$

Both integrals are real valued, so we can read off the real and imaginary parts. On the critical line ($u = 0$), the imaginary part is zero (was known). Implications for the graph...worth plotting!

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If a zero is not simple, then both expressions above have to be zero: unlikely, unless $v = 0$ (the critical point), where they are sometimes not simple.

Last words

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but there is no time left.

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Thank you!