ODE : Midterm 2

Justify all answers. Without proper justification, you may not get any credit. To ensure full credit, also draw a box around the answer(s) to each problem. In any problem, you may use the result of a previous problem without repeating the work. If your answer to a problem depends on an earlier problem that you could not solve, you can write what you would have done if you could have solved the earlier problem. When you use the table of Laplace transforms, it is recommended (though not required) that you mention the number of the formula that you use (e.g., “By 5, ...”). The exam is out of 30 points. Good luck!

1. (3 points) Let

\[ f(t) = \begin{cases} 
1 & 0 \leq t < 6 \\
\tan t & 6 \leq t < 10 \\
e^t & 10 \leq t 
\end{cases} \]

Express \( f(t) \) in terms of unit step functions \( u_c(t) \).

2. (3 points) Find the Laplace transform of \((t - 7)u_4(t)\).

3. (4 points) Find the inverse Laplace transform of \( F(s) = \frac{1}{s^2(s+1)} \).

4. (3 points) Find the inverse Laplace transform of \( F(s) = \frac{s}{s^2 + 2s + 2} \).

5. (3 points) Find the inverse Laplace transform of \( F(s) = e^{-2s} \frac{s}{s^2 + 2s + 2} \). Your answer should be an explicit function that should involve only standard functions (such as trigonometric functions, unit step function, etc.).

6. (5 points) Consider the initial value problem \( y'' + 2y' + 3y = 3u_2(t); \ y(0) = 1, y'(0) = 0 \). Find the Laplace transform of the solution \( y(t) \) (i.e., start solving the problem using Laplace transforms and stop after you find \( Y(s) \); if in doubt, you may solve the problem itself).

7. (3 points) Consider the initial value problem \( y'' + 2y' + 3y = 2\delta(t-3); \ y(0) = 1, y'(0) = 0 \) (Note: only the right side of the ODE is different from that in the previous problem; the initial conditions are the same). Find the Laplace transform of the solution \( y(t) \) (i.e., start solving the problem using Laplace transforms and stop after you find \( Y(s) \); if in doubt, you may solve the problem itself).

8. (6 points) Consider the differential equation \( y'' + 2xy' + y = 0 \). Seek a power series solution to the differential equation about the point \( x_0 = 0 \) and find the recurrence relation; you do NOT need to write down any terms of the series solution.