Calabi Yau threefolds from singular plane curves

Anatoly Libgober

University of Illinois at Chicago
Chicago, Illinois

AMS Special session on Singularities and Physics, Knoxville.
March 21-23, 2014
Elliptic fibrations corresponding to plane curves
A. Libgober (Department of Mathematics)  Calabi Yau threefolds from singular plane curves  Colloquium  2 / 30

Content

1. Elliptic fibrations corresponding to plane curves

2. Fundamental group of the complements to plane singular curves
Content

1. Elliptic fibrations corresponding to plane curves
2. Fundamental group of the complements to plane singular curves
3. Mordell-Weil ranks of elliptic fibrations corresponding to plane curves
Content

1. Elliptic fibrations corresponding to plane curves
2. Fundamental group of the complements to plane singular curves
3. Mordell-Weil ranks of elliptic fibrations corresponding to plane curves
4. Construction of singular curves with superabundant sets of cusps
Content

1. Elliptic fibrations corresponding to plane curves
2. Fundamental group of the complements to plane singular curves
3. Mordell-Weil ranks of elliptic fibrations corresponding to plane curves
4. Construction of singular curves with superabundant sets of cusps
5. Mordell-Weil ranks of families of abelian varieties
Let $f(x, y) = 0$ be equation of plane curve and let

$$F : u^2 + v^3 = f(x, y)$$

be a hypersurface in $\mathbb{C}^4$. (Weierstrass model with one of the parameters vanishing).
Let $f(x, y) = 0$ be equation of plane curve and let

$$F : u^2 + v^3 = f(x, y)$$

be a hypersurface in $\mathbb{C}^4$. (Weirstrass model with one of the parameters vanishing).

**Definition**

A projective birational model $X_f$ (possibly singular) of $F$ will be called an elliptic fibration corresponding to curve $f$. We shall assume that one has holomorphic map $X_f \rightarrow \mathbb{P}^2$. 
Let $f(x, y) = 0$ be equation of plane curve and let

$$F : u^2 + v^3 = f(x, y)$$

be a hypersurface in $\mathbb{C}^4$. (Weirstrass model with one of the parameters vanishing).

**Definition**

A projective birational model $X_f$ (possibly singular) of $F$ will be called an elliptic fibration corresponding to curve $f$. We shall assume that one has holomorphic map $X_f \to \mathbb{P}^2$.

One can view $X_f$ as an elliptic curve over $\mathbb{C}(x, y)$. Zero of this elliptic curve is the point at infinity. Discriminant is $f(x, y)$. 
$X_f$ is isotrivial: fibers are elliptic curves with $j = 0$. It becomes trivial over field $\mathbb{C}(x, y)(f(x, y)^{\frac{1}{6}})$: change variables $u \rightarrow u \cdot f^\frac{1}{2}$, $v \rightarrow v \cdot f^\frac{1}{3}$ yields:

$$u^2 + v^3 = f(x, y) \Rightarrow u^2 + v^3 = 1$$

Pullback of this fibration to $V : z^6 = f(x, y)$ is trivial.
Biregular model: If degree of $f$ is $6n$ than a possible $X_f$ is a hypersurface in weighted projective space

$$\mathbb{P}^4(3n, 2n, 1, 1, 1)$$

given by $u^2 + v^3 = \tilde{f}(x, y, z)$, ($\tilde{f}$ is a homogenized $f$).
Biregular model: If degree of $f$ is $6n$ than a possible $X_f$ is a hypersurface in weighted projective space

$$\mathbb{P}^4(3n, 2n, 1, 1, 1)$$

given by $u^2 + v^3 = \bar{f}(x, y, z)$, ($\bar{f}$ is a homogenized $f$).

If $n = 3$ then $X_f$ is Calabi Yau elliptic threefold. If $f(x, y)$ is non-generic but has singularities which are not too bad e.g. locally given by $x^2 = y^3$ (cusps) or $x^2 = y^2$ (nodes) then $X_f$ has terminal singularities.
Definition

\( \mathbb{C}(x, y) \)-point of \( X_f \) is a section \( \mathbb{C}^2 \rightarrow X_f \) defined on a Zariski open subset of \( \mathbb{C}^2 \). If \( K \) is a finite extension of \( \mathbb{C}(x, y) \) i.e. a field of functions on a ramified covering space \( V \rightarrow \mathbb{C}^2 \), then \( K \)-point is section \( V \times_{\mathbb{C}^2} X_f \).
**Definition**

$\mathbb{C}(x, y)$-point of $X_f$ is a section $\mathbb{C}^2 \to X_f$ defined on a Zariski open subset of $\mathbb{C}^2$. If $K$ is a finite extension of $\mathbb{C}(x, y)$ i.e. a field of functions on a ramified covering space $V \to \mathbb{C}^2$, then $K$-point is section $V \times_{\mathbb{C}^2} X_f$.

**Warning:** Group of $K$-points of $X_f$ is infinitely generated if curve over a function field $K$ is trivial but one has the following theorem:
Definition

\( \mathbb{C}(x, y) \)-point of \( X_f \) is a section \( \mathbb{C}^2 \to X_f \) defined on a Zariski open subset of \( \mathbb{C}^2 \). If \( K \) is a finite extension of \( \mathbb{C}(x, y) \) i.e. a field of functions on a ramified covering space \( V \to \mathbb{C}^2 \), then \( K \)-point is section \( V \times_{\mathbb{C}^2} X_f \).

Warning: Group of \( K \)-points of \( X_f \) is infinitely generated if curve over a function field \( K \) is trivial but one has the following theorem:

Theorem

(\text{Lang Neron}). If \( E_K \) is an elliptic curve over a function field \( K \) which is a transcendental extension of \( \mathbb{C} \) then group of \( K \) points contains a subgroup \( \text{Chowtr}(E_K) \) of points over \( \mathbb{C} \) of \( E_K \) such that quotient of the group of \( K \)-points by \( \text{Chowtr}(E_K) \) is \textbf{finitely generated} (Mordell-Weil group of \( E_K \)).
Problem:

What are possible ranks of $\text{MW}(X_f)$? More general question: Given a field $K$ for which one knows that $\text{MW}$ is finitely generated, what are possible ranks of elliptic curves over $K$?
Problem:

What are possible ranks of $\text{MW}(X_f)$? More general question: Given a field $K$ for which one knows that $\text{MW}$ is finitely generated, what are possible ranks of elliptic curves over $K$?

1. $K$ is a number field: open (record is 28 (N. Elkies)).
2. $K = \mathbb{F}(t)$: ranks are unbounded (Shafarevich-Tate, 1968 in isotrivial case, D. Ulmer 2002 for non-isotrivial elliptic curves).
3. $K = \mathbb{C}(t)$: open (record is 68 for $u^2 + v^3 = t^{360} - 1$ (Shioda)).
Problem:

What are possible ranks of $\text{MW}(X_f)$? More general question: Given a field $K$ for which one knows that $\text{MW}$ is finitely generated, what are possible ranks of elliptic curves over $K$?

1. $K$ is a number field: open (record is 28 (N. Elkies)).
2. $K = \mathbb{F}(t)$: ranks are unbounded (Shafarevich-Tate, 1968 in isotrivial case, D. Ulmer 2002 for non-isotrivial elliptic curves).
3. $K = \mathbb{C}(t)$: open (record is 68 for $u^2 + v^3 = t^{360} - 1$ (Shioda)).

Main results: $\text{MW}(X_f)$ can be expressed 1) in terms of fundamental group of the complement to $C$ more precisely $\pi'_1 / \pi''_1$
2) Geometry (position) of the set of cusps
There is a relation between distribution of Mordell Weil ranks and topological invariants of plane singular curves.
Key tool is study of geometry of cyclic covers. Let $C$ be irreducible plane curve and $G = \pi_1(\mathbb{P}^2 \setminus C)$. There is correspondence between

$$\text{rk}G' / G'' \otimes \mathbb{Q}$$

and

$$\text{rk}H_1(V_{\deg C})$$

for curves with arbitrary singularities. If $C$ is irreducible then $\text{rk}G' / G'' = \dim H^1(V_{\deg C}, \mathcal{C})$ where $V_{\deg C} : z^{\deg C} = f(x, y)$. 
In general case, one can express homology of cyclic covers in terms of Alexander polynomial. Let $G$: let $lk : G \to \mathbb{Z}_{\deg C}$ be linking number modulo $\deg C$. Let $G' = Ker lk$. Then

$$0 \to G'/G'' \to G/G'' \to \mathbb{Z}_{\deg C} \to 0$$

Generator of $\mathbb{Z}_k$ acts on $G'/G'' \otimes \mathbb{C}$ and characteristic polynomial $\Delta_C(t)$ is the **Alexander polynomial** of $C$. 
Theorem

(Libgober, 1982) 1. \( rkH^1(V_d, \mathcal{C}) \) is the number of common roots for \( \Delta_C(t) \) and \( t^d - 1 \).
2. If \( \Delta_C(t) \mid \prod_{P \in \text{Sing}_C} \Delta_P(t) \).

In particular if \( C \) has nodes and cusps then

\[
\Delta_C(t) = (t^2 - t + 1)^s, \quad s \geq 0
\]

Moreover, \( \Delta_C(t) = 1 \) unless \( 6 \mid \text{deg}(C) \).
Theorem

Let $\xi_P^i \in \mathbb{Q}$ be set of rational numbers such that $\exp 2\pi i \xi_P^i$ is a root of Alexander polynomial $\Delta_P(t)$. There is ideal sheaf $\mathcal{I}_{\xi_P, C}$ such that $O_{\mathbb{P}^2}/\mathcal{I}_{\xi_P, C}$ is supported at singularities of $C$ and such that multiplicity of the root $\xi_P$ in Alexander polynomials is $H^1(\mathbb{P}^2, \mathcal{I}_{\xi_P}(d_\xi_P))$.

In particular if $C$ has nodes and cusps only

$$\Delta_C(t) = (t^2 - t + 1)^s \text{ where } s = \dim H^1(\mathbb{P}^2, \mathcal{I}_{\text{cusps}}(\frac{5d}{6} - 3)) =$$

$$\dim H^0(\mathbb{P}^2, \mathcal{I}_{\text{cusps}}(\frac{5d}{6} - 3)) - \chi(\mathcal{I}_{\text{cusps}}(\frac{5d}{6} - 3))$$
Theorem

(Cogolludo-Libgober) Let $X_f$ be elliptic threefold $u^2 + v^3 = f(x, y)$. Let $s$ be the multiplicity of the root $\exp(\frac{2\pi i}{6})$. Then

$$rkMW(X_f) = 2s$$
Theorem

(Cogolludo-Libgober) Let $X_f$ be elliptic threefold $u^2 + v^3 = f(x, y)$. Let $s$ be the multiplicity of the root $\exp(\frac{2\pi i}{6})$. Then

$$\text{rkMW}(X_f) = 2s$$

Remark

Special case when $\text{deg}(f) = 6$ this coincides with result of Hulek-Kloosterman who enumerated Mordell Weil ranks of elliptic threefolds corresponding to sextics with various number of cusps.
Theorem

(Cogolludo-Libgober) Let $X_f$ be elliptic threefold $u^2 + v^3 = f(x, y)$. Let $s$ be the multiplicity of the root $\exp(\frac{2\pi i}{6})$. Then

$$\text{rkMW}(X_f) = 2s$$

Remark

Special case when $\text{deg}(f) = 6$ this coincides with result of Hulek-Kloosterman who enumerated Mordell Weil ranks of elliptic threefolds corresponding to sextics with various number of cusps.

Corollary

$$\text{rkMW}(X_f) = 2\dim H^1(P^2, \mathcal{I}_{\text{cusps}}(\frac{5\text{deg}(f)}{6} - 3))$$

If $C$ is smooth or has not enough cusps to make them superabundant then $\text{rkMW}_f = 0$. 
Remark

(on the proof): Since $X_f$ trivializes over $V_f : z^6 = f(x, y)$ is is enough to describe $\mathbb{Z}_6$-invariant points in $V_f \times E_0$ or equivalently $\text{Mor}_{\mathbb{Z}_6}(V_f, E_0)$. 
Remark

(on the proof): Since $X_f$ trivializes over $V_f : z^6 = f(x, y)$ is is enough to describe $\mathbb{Z}_6$-invariant points in $V_f \times E_0$ or equivalently $\text{Mor}_{\mathbb{Z}_6}(V_f, E_0)$.

In fact $\text{Mor}(V_f, E_0) = \text{Hom}(\text{Alb}(V_f), E_0)$ where

$$\text{Alb}(V_f) = H^0(\Omega^1_{V_f})^*/H_1(V_f, \mathbb{Z})$$
Remark

(on the proof): Since $X_f$ trivializes over $V_f : z^6 = f(x, y)$ is is enough to describe $\mathbb{Z}_6$-invariant points in $V_f \times E_0$ or equivalently $\text{Mor}_{\mathbb{Z}_6}(V_f, E_0)$.

In fact $\text{Mor}(V_f, E_0) = \text{Hom}(\text{Alb}(V_f), E_0)$ where

$$\text{Alb}(V_f) = H^0(\Omega^1_{V_f})^*/H_1(V_f, \mathbb{Z})$$

Theorem

(Cogolludo-Libgober)

$$\text{Alb}(V_f) = E_0^s$$
Remark

Albanese variety of cyclic covers of $\mathbb{P}^1$ rarely splits. One case of Albanese of 3-fold cover of plane was obtained classically by Comessatti (arrangement of 9 lines dual to inflection points of a plane elliptic curve for which albanese of 3-fold cyclic cover is biregular to $E_0^2$).
Theorem

(Cogolludo-Libgober) Bounds on the growth $MW(X_f)$. For a curve $f = 0$ of degree $d$

$$rkMW(X_f) \leq \frac{5}{3}d - 2$$
**Theorem**

*(Cogolludo-Libgober)* **Bounds on the growth** $MW(X_f)$. For a curve $f = 0$ of degree $d$

$$rkMW(X_f) \leq \frac{5}{3}d - 2$$

**Theorem**

*(Kloosterman)* **For a curve** $f = 0$ of degree $6k$ the upper bound on $rkMW(X_f)$ is

$$\frac{1}{18}(125 + \sqrt{73} - \sqrt{2302 - 106\sqrt{73}})k + O(1)$$
Exact bound on the maximal number of cusps $\kappa(d)$ of a curve of degree $d$ is unknown but (Hirano-Myaoka-Yau-Langer):

$$
\frac{9}{32} \leq \limsup_{d \to \infty} \frac{\kappa(d)}{d^2} \leq \frac{125 + \sqrt{73}}{432} < \frac{5}{16}
$$
Exact bound on the maximal number of cusps $\kappa(d)$ of a curve of degree $d$ is unknown but (Hirano-Myaoka-Yau-Langer):

$$\frac{9}{32} \leq \limsup_{d \to \infty} \frac{\kappa(d)}{d^2} \leq \frac{125 + \sqrt{73}}{432} < \frac{5}{16}$$

Sometimes lower degree term helps to get better bounds e.g. (Hirzebruch-Ivinskis):

$$\kappa(d) \leq \frac{5}{16} d^2 - \frac{3}{8} d$$
Cuspidal sextics:
Cusps estimate $\kappa(6) \leq 11$ but actual value $\kappa(6) = 9$. Curves of degree 6: Alexander polynomial is $(t^2 - t + 1)^s$ where $s = 1, 2, 3$. $s = 1, 2, 3$ for curves with 7, 8, 9 cusps. For $\kappa(C) = 6$, $s = 1$ or $s = 0$ depending on whether 6 cusps are on conic or not ($s = \dim H^1(I_{cusp}(2))$, ($\frac{5d}{6} - 3 = 2$)).
In particular, possible Mordell-Weil ranks of these Fano elliptically fibered threefolds are 0, 1, 2, 3.
Curves of degree 12. $\kappa(12) = 44$. There exist curve of with 39 cusps such that $s(C) = 4$ (Cogolludo-Libgober). Construction: Let $D$ be curve of degree 4 with 3 cusps (dual to cubic with one node). It has one bitangent. Consider coordinate system in $\mathbb{P}^2$ in which $x = 0$ is bitangent and $y = 0, z = 0$ are ordinary tangents to $D$. 
Curves of degree 12. $\kappa(12) = 44$. There exist curves of with 39 cusps such that $s(C) = 4$ (Cogolludo-Libgober). Construction: Let $D$ be curve of degree 4 with 3 cusps (dual to cubic with one node). It has one bitangent. Consider coordinate system in $\mathbb{P}^2$ in which $x = 0$ is bitangent and $y = 0, z = 0$ are ordinary tangents to $D$.

Consider Kummer cover $\pi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(x, y, z) \to (x^3, y^3, z^3)$ and $\pi^{-1}(D)$.

3 Cusps of $D \to 3 \times 9 = 27$ Cusps of $C$
Curves of degree 12. $\kappa(12) = 44$. There exist curve of with 39 cusps such that $s(C) = 4$ (Cogolludo-Libgober). Construction: Let $D$ be curve of degree 4 with 3 cusps (dual to cubic with one node). It has one bitangent. Consider coordinate system in $\mathbb{P}^2$ in which $x = 0$ is bitangent and $y = 0, z = 0$ are ordinary tangents to $D$.

Consider Kummer cover $\pi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(x, y, z) \to (x^3, y^3, z^3)$ and $\pi^{-1}(D)$.

3 Cusps of $D \to 3 \times 9 = 27$ Cusps of $C$

Each tangency point of $D$ to ramification locus of Kummer cover gives 3 cusps of $D$. Hence one get 39 cusps of $D$. 
Theorem

(Cogolludo-Libgober) For this curve $s = \dim H^1(\mathbb{P}^2, \mathcal{I}_{\text{cusps}}(7)) = 4$ and hence Mordell Weil rank of $X_f$ is 4.
Construction of singular curves with superabundant sets of cusps

Theorem (Cogolludo-Libgober) For this curve \( s = \dim H^1(\mathbb{P}^2, \mathcal{I}_{\text{cusps}}(7)) = 4 \) and hence Mordell Weil rank of \( X_f \) is 4.

Theorem (Libgober) There is curve of degree 18 with 90 cusps for which superabundance of the set of cusps (for the curves of degree 12) is 3. It gives rise to CY elliptic threefold with \( \text{rkMW}(X_f) = 3 \).

Remark 1. Upper bound for the number of cusps is 99.

Remark 2. The curve is preimage of sextic with 9 cusps in Kummer cover (using cooredinates axis tangent to the curve; sextic with 9 cusps has no bitangents).
Theorem (Cogolludo-Libgober) For this curve $s = \dim H^1(\mathbb{P}^2, \mathcal{I}_{\text{cusps}}(7)) = 4$ and hence Mordell Weil rank of $X_f$ is 4.

Theorem (Libgober) There is curve of degree 18 with 90 cusps for which superabundance of the set of cusps (for the curves of degree 12) is 3. It gives rise to CY elliptic threefold with $\text{rkMW}(X_f) = 3$.

Remark
1. Upper bound for the number of cusps is 99.
2. The curve is preimage of sextic with 9 cusps in Kummer cover (using coorordinates axis tangent to the curve; sextic with 9 cusps has no bitangents).
Curves with more complicated than ordinary cusps appear as discriminants of families of \textbf{abelian varieties} and Mordell-Weil ranks of such (iso-trivial) abelian varieties over function fields can be related to topology of the complement to this discriminant.
Easy way to construct such families of abelian varieties is to look at families of Jacobians: Consider Jacobian of the curve $X_f$ over $\mathbb{C}(x, y)$ given in $(u, v)$ plane by the equation (i.e. $f = (x^p + y^p)^2 + (y^2 + 1)^p$)

$$u^p = v^2 + (x^p + y^p)^2 + (y^2 + 1)^p$$

Then one has $\text{rkMW}(\text{Jac}(X_f)) = p - 1$
Easy way to construct such families of abelian varieties is to look at families of Jacobians: Consider Jacobian of the curve $X_f$ over $\mathbb{C}(x, y)$ given in $(u, v)$ plane by the equation (i.e. $f = (x^p + y^p)^2 + (y^2 + 1)^p$)

$$u^p = v^2 + (x^p + y^p)^2 + (y^2 + 1)^p$$

Then one has $rkMW(Jac(X_f)) = p - 1$

Note: the Jacobian of generic fiber of the family is a simple abelian variety; For $p = 5$ one has CY threefold with terminal singularities with pencil of genus 2 curves)
Theorem

Let $A$ be an isotrivial abelian variety over field $\mathbf{C}(x,y)$ i.e. we have morphism $\pi : A \to \mathbf{P}^2$ such that generic fiber $A$ is an abelian variety over $\mathbf{C}$.

Let $\Delta \subset \mathbf{P}^2$ be the discriminant of $\pi$ and let $G \subset \text{Aut}A$ be the holonomy group of $A$.

Assume that:

a) $G$ is a cyclic group of order $d$ acting on generic fiber $A$ of $\pi$ without fixed subvarieties of a positive dimension.

b) The singularities of $\Delta$ have CM type and $\Delta$ is irreducible.

Then
1. The rank of the Mordell-Weil group of $\mathcal{A}$ is zero, unless the generic fiber of $\pi$ is an abelian variety of CM-type with endomorphism algebra containing a cyclotomic field.

2. Assume that the generic fiber $\mathcal{A}$ of $\pi$ is a simple abelian variety of CM type corresponding to the field $\mathbb{Q}(\zeta_d)$. Let $s$ be the multiplicity of the factor $\Phi_d(t)$ of the Alexander polynomial of $\pi_1(\mathbb{P}^2 - \Delta)$ where $\Phi_d(t)$ is the cyclotomic polynomial of degree $d$. Then:

$$\text{rk}MW(\mathcal{A}, \mathbb{C}(x, y)) \leq s \cdot \phi(d) \quad (*)$$

(here $\phi(d) = \text{deg} \Phi_d(t)$ is the Euler function).

3. Let $\mathcal{A}$ be an abelian variety as in 2. If $d$ is the order of the holonomy of $\mathcal{A}$ and the Albanese variety $\text{Alb}(X_d)$ of the $d$-fold cover $X_d$ of $X$ ramified over $\Delta$ has $\mathcal{A}$ as its direct summand with multiplicity $s$ then one has equality in ($*$).
Abelian varieties of CM-type

1. CM field is an imaginary quadratic extension of a totally real number field.

2. For a CM field $K$ of degree $g$ over $\mathbb{Q}$, a CM type $\Phi$ is a collection of pairwise, not conjugate, embeddings $\sigma_1, \ldots, \sigma_g$ of $K \to \mathbb{C}$.

3. For a CM field $K$ with a chosen CM-type $\Phi$ and a lattice in $K$ i.e. a finitely generated subgroup $\Lambda$ such that $E = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ (e.g. a ring of integers in $K$) corresponds the torus $K \otimes_{\mathbb{Q}} \mathbb{R}/\Lambda$ with the complex structure induced from the identification $K \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}^{\dim_{\mathbb{Q}} E/2}$ given by the direct sum of the homomorphisms $\phi \in \Phi$ where $\Phi$ is the CM type. This complex torus is an abelian variety.
Example

Let $p$ be an odd prime. Consider the set $\Phi$ of roots of unity of degree $p$ with positive imaginary part (i.e. $\exp\left(\frac{2\pi i k}{p}\right)$ for $1 \leq k \leq \frac{p-1}{2}$). The set of embeddings of $\mathbb{Q}(\zeta_p)$ induced by the maps $\exp\left(\frac{2\pi i}{p}\right) \to \omega, \omega \in \Phi$ provides a CM type of $\mathbb{Q}(\zeta_p)$. Note that this CM type is primitive in the sense that the corresponding abelian variety is simple.
**Definition**

**Local Albanese varieties and Singularities of CM type**

1. Given a pure Hodge structure \((H_Z, F)\) of weight \(-1\), one associates to it a complex torus as follows

\[
A_H = H_Z \backslash H_C / F^0 H_C
\]

In the case when the Hodge structure is polarized, \(A_H\) is an abelian variety.

2. Local Albanese variety \(Alb_\phi\) of a plane curve singularity \(\phi(x, y) = 0\) is the abelian variety corresponding to the Hodge structure on homology \(H_1(M_f, \mathbb{Z})\) of the Milnor fiber which is dual to the canonical cohomological limit mixed Hodge structure on the cohomology of a Milnor fiber.
3. A plane curve singularity is called a singularity of CM type if its local Albanese variety is isogenous to a product of simple abelian varieties of CM type.
Example

For curve singularities with one characteristic pair given by \( x^p = y^q \) where \( \gcd(p, q) = 1 \), the number of eigenvalues of the monodromy acting on \( \text{Gr}_F^0 H^1(M) \) (\( M \) is the Milnor fiber) is equal to \( \frac{(p-1)(q-1)}{2} \). More precisely, the action on \( H_1(M) \) is semi-simple and has as the characteristic polynomial

\[
\Delta_{p,q} = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}
\]
Example

The characteristic polynomial of the action on $Gr^0_F H^1(M)$ is

$$\Pi(t - \exp(-2\pi \sqrt{-1} \alpha)),$$

where

$$\alpha = \frac{i}{p} + \frac{j}{q}, \quad 0 < \alpha < 1, \quad 0 < i < p, \quad 0 < j < q$$

In particular for $f(x, y) = x^2 + y^3$ the only eigenvalue on $F^0$ is $\exp(\frac{2\pi \sqrt{-1}}{6})$. More generally, for the singularity $x^2 + y^p$ where $p$ is an odd prime, the field generated by the eigenvalues of the monodromy is $\mathbb{Q}(\zeta_p)$ and the CM type corresponds to subset set $\exp(\frac{2\pi \sqrt{-1} j}{p})$ where $\frac{1}{2} + \frac{i}{p} < 1$ i.e. coincides with the CM type discussed in example.
Remark

1. For $p = 5$ one has Jacobian of hyperelliptic curve of genus 2: $u^2 = v^5 + c$

2. These examples show that $rkw$ ranks can be arbitrary large for isotrivial families of abelian varieties but there is interesting conjecture about distribution and boundness of ranks. In terms of Alexander polynomials this is the question of boundness of multiplicity of a fixed cyclotomic factor (e.g. $t^2 - t + 1$, $\frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$, etc. in Alexander polynomials of plane curves.)