Hardy-Littlewood Inequality for Quasiregular Maps on Carnot Groups

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ABSTRACT. We generalize a Hardy-Littlewood inequality for L^p-norms of conjugate harmonic functions to horizontal components of quasiregular mappings on Carnot groups.

1. Introduction

In [1], Hardy and Littlewood proved the following result.

Theorem 1.1 *If $u+iv$ is analytic in a disk D centered at z_0, then there exists a constant C, depending only on p, $0 < p$, such that*

$$\int_D |u - u(z_0)|^p \, dxdy \leq C \int_D |v|^p \, dxdy. \tag{1}$$

A natural generalization of analytic functions to n-dimensional Euclidean space are quasiregular mappings.(See [2] and [3].) An analogue of Theorem 1.1 for quasiregular mappings in John domains in Euclidean space appeared in [4]. Recently the analytical tools used in the proof of this result have been generalized to Carnot groups. We give an account of some of these advances and obtain an analogue of Theorem 1.1 in this context. A Carnot group is a connected, simply connected, nilpotent Lie group G of topological $\dim G = N \geq 2$ equipped with a graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ so that $[V_1, V_i] = V_{i+1}$ for i=1,2,...,r-1 and $[V_1, V_r] = 0$. This defines an r-step Carnot group. As usual, elements of $\mathcal{G}$ will be identified with left-invariant vectors fields on G. We adopt when possible the elegant notation from [5]. We fix a left-invariant Riemannian metric g on G with $g(X_i, X_j) = \delta_{ij}$. We denote the inner product with respect to this metric, as well as all other inner products, by $\langle , \rangle$. We assume that $\dim V_1 = m \geq 2$ and fix an orthonormal basis of $V_1 : X_1, X_2, ..., X_m$. The horizontal tangent bundle of G, HT , is the subbundle determined by V_1 with horizontal tangent space HT_x the fiber span$[X_1(x), ..., X_m(x)]$. We use a fixed global coordinate system as $\exp : \mathcal{G} \to G$ is a diffeomorphism (since G is simply-connected and nilpotent). We extend $X_1, ..., X_m$ to an orthonormal basis $X_1, ..., X_m, T_1, ..., T_{N-m}$ of $\mathcal{G}$. All integrals will be with respect to the bi-invariant Harr measure on G which arises as the push-forward of the Lebesque measure in $\mathbb{R}^N$ under the exponential map. We denote by $|E|$ the measure of a measurable set E.

We normalize the Harr measure so that the measure of the unit ball is one. We denote by Q the homogeneous dimension of the Carnot group G defined by $Q = \sum_{i=1}^{r} i \, \text{dim} V_i$. The dual basis of $\mathcal{G}$ is denoted by dx_1,...,dx_m,τ_1,...,τ_{N-m} with the pairing $\langle , \rangle : T_x \times T_x^* \to \mathbb{R}$. Notice $\langle X_j, dx_i \rangle = \langle T_j, dx_i \rangle = \delta_{ij}$, and $\langle X_j, \tau_i \rangle = \langle T_j, dx_i \rangle = 0$. Also a vector field X is called horizontal if $\langle X, \tau_i \rangle = 0$ for all $i = 1, ..., N-m$. We write $|v|^2 = \langle v, v \rangle$, d for the distributional exterior derivative and δ for the codifferential adjoint. We use the following spaces where U is an open set in G :

$\Lambda^k T$: k-forms in the tangent bundle of G,

$C_0^\infty(U, \Lambda^k)$: infinitely differentiable compactly supported k-forms in U,

$L^p(U, \Lambda^k)$: k-forms with coefficients in $L^p(U)$, $p > 0$,

$W^{1,p}(U, \Lambda^k)$: Sobolev space of k-forms $u \in L^p(U, \Lambda^k)$ such that $du \in L^p(U, \Lambda^{k-1})$,

$HW^{1,p}(U)$: horizontal Sobolev space of functions $u \in L^p(U)$ such that the distributional derivatives $X_i u \in L^p(U)$ for i = 1,...,m.

When u is in the local horizontal Sobolev space $HW^{1,p}_{loc}(U)$ we write the horizontal differential as $d_0 u = X_1 u dx_1 + ... + X_m u dx_m$. (The horizontal gradient $\nabla_0 u = X_1 u X_1 + ... + X_m u X_m$ appears in the literature. Notice that $|d_0 u| = |\nabla_0 u|$.) The family of dilations on G, $\{\delta_t : t > 0\}$, is the lift to G of the automorphism δ_t of $\mathcal{G}$ which acts on each V_i by multiplication by t^i. A path in G is called horizontal if its tangents lie in V_1. The (left-invariant) Carnot-Carathéodory distance , $d_c(x, y)$, between x and y is the infimum of the lengths, measured in the Riemannian metric g, of all horizontal paths which join x to y. A homogeneous norm is given by $|x| = d_c(0, x)$. We have $|\delta_t(x)| = t|x|$. We write $B_r(x) = \{y \in G : |x^{-1}y| < r\}$ for the ball centered at x of radius r. Since the Jacobian determinant of the dilatation δ_r is r^Q and we have normalized the measure, $|B_r| = r^Q$. For $\sigma > 0$, we write σB for the ball with the same center as B and radius σ times that of B. As references for Carnot groups we mention [6],[7],[8] and [9].

Example 1.2 *Euclidean space $\mathbb{R}^n$ with its usual Abelian group structure is a Carnot group. Here $Q = n$ and $X_i = \partial/\partial x_i$.*

Example 1.3 *Each Heisenberg group H_n, $n \geq 1$, is homeomorphic to $\mathbb{R}^{2n+1}$. They form a family of noncomutative Carnot groups which arise as the nilpotent part of the Iwasawa decomposition of $U(n,1)$, the isometry group of the complex n-dimensional hyperbolic space. Denoting points in H_n by (z,t) with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$ we have the group law given as*

$$(z, t) \circ (z', t') = (z + z', t + t' + 2 \sum_{j=1}^{n} Im(z_j \bar{z}'_j)). \tag{2}$$

With the notation $z_j = x_j + iy_j$, the horizontal space V_1 is spanned by the basis

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \tag{3}$$

$$Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}. \tag{4}$$

The one dimensional center V_2 is spanned by the vector field $T = \partial/\partial t$ with commutator relations $[X_j, Y_j] = -4T$. All other brackets of $\{X_1, Y_1, ..., X_n, Y_n\}$ are zero. The *homogeneous dimension of* H_n *is* $Q = 2n + 2$. *A homogeneous norm is given by*

$$|(z,t)| = (|z|^4 + t^2)^{1/4}. \tag{5}$$

Example 1.4 *A Generalized Heisenberg group, or H-type group, is a Carnot group with a two-step Lie algebra* $\mathcal{G} = V_1 \oplus V_2$ *and an inner product* $\langle , \rangle$ *in* $\mathcal{G}$ *such that the linear map* $J : V_2 \to EndV_1$ *defined by the condition*

$$\langle J_z(u), v\rangle = \langle z, [u,v]\rangle, \tag{6}$$

satifies

$$J_z^2 = -\langle z, z\rangle \mathbf{Id} \tag{7}$$

for all $z \in V_2$. *For each* $x \in G$, *let* $v(x) \in V_1$ *and* $z(x) \in V_2$ *be such that* $x = \exp(v(x) + z(x))$. *Then*

$$|x| = (|v(x)|^4 + 16|z(x)|^2)^{1/4} \tag{8}$$

defines a homogeneous norm in G. *For each* $m \in \mathbb{N}$ *there exist infinitely many generalized Heisenberg groups with* $dimV_2 = m$. *These include the nilpotent groups in the Iwasawa decomposition of the simple rank-one groups* $SO(n,1), SU(n,1), Sp(n,1)$ *and* F_4^{-20}.

See [10] for material about these groups.

2. Subelliptic equations

We use the fact that the horizontal components of quasiregular mappings on Carnot groups satisfy a certain nonlinear subelliptic equation.

Definition 2.1 *An, a.e.* $x \in G$ *continuous map* $A_x : \Lambda^1 HT_x \to \Lambda^1 HT_x$ *is a Carathéodory function if*

$$x \mapsto A_x(\omega(x)) \tag{9}$$

is a measureable section of the horizontal cotangent bundle $HT^* = \Lambda^1 HT$ *whenever* ω *is and there exists* $1 < \alpha < \infty$ *and* $\nu > 0$ *such that*

$$\langle A_x(\xi), \xi\rangle \geq \nu^{-1}\langle \xi, \xi\rangle^{\alpha/2} \tag{10}$$

and

$$|A_x(\xi)| \leq \nu|\xi|^{\alpha-1} \tag{11}$$

for a.e. $x \in G$ *and all* $\xi \in \Lambda^1 HT_x$.

Given such a map A_x ,we say that a function $u \in HW^{1,\alpha}_{loc}(U)$ is a solution to

$$\delta A_x(d_0 u) = 0 \tag{12}$$

in U if

$$\int_U \langle A_x(d_0 u), d_0 \eta \rangle = 0 \tag{13}$$

for all $\eta \in C_0^\infty(U)$.

For example when $A_x(\xi) = |\xi|^{p-2}\xi$, (12) is the sub-p-harmonic equation

$$\delta(|d_0|^{p-2} d_0 u) = 0 \tag{14}$$

which is the sublaplacian $\sum_{i=1}^m X_i^2 u = 0$ when p = 2. A solution, u, to (2.1) enjoys the following three properties. The first in a version of a Poincaré-Sobolev inequality found in [12] (see also [13]): For $0 < p < \infty$ there exists a constant C_1 such that

$$\left(\frac{1}{|B_r|} \int_{B_r} |u - u_{B_r}|^p \right)^{1/p} \leq C_1 r \left(\frac{1}{|B_r|} \int_{B_r} |d_0 u|^p \right)^{1/p} . \tag{15}$$

Second is a Caccioppoli estimate (see [5]) : There exists a constant C_2 such that, for the exponent α in Definition 2.1,

$$\left(\frac{1}{|B_r|} \int_{B_r} |d_0 u|^\alpha \right)^{1/\alpha} \leq \frac{C_2}{r} \left(\frac{1}{|B_r|} \int_{2B_r} |u|^\alpha \right)^{1/\alpha} . \tag{16}$$

Third are the so-called weak reverse Hölder inequalities (see [5] and [4]) : For $0 < s, t < \infty$, there exists a constant C_3 such that

$$\left(\frac{1}{|B_r|} \int_{B_r} |u|^s \right)^{1/s} \leq C_3 \left(\frac{1}{|B_r|} \int_{2B_r} |u|^t \right)^{1/t} . \tag{17}$$

3. Quasiregular mappings

Definition 3.1 *Suppose that $f : U \to G$ is in $W^{1,1}_{loc}(U)$ (so that the formal derivative $f_* : T_x \to T_{f(x)}$ exists a.e.). We say that f is a (generalized) contact map if*

$$f_* V_1 \subset V_1 \tag{18}$$

a.e. (So for each $X \in HT_x, f_*(x)X \in HT_{f(x)}$ a.e.)

Writing $\tau = \tau_1 \wedge \cdots \wedge \tau_{N-m}$ and $f^*\tau$ for the pullback of the form τ under f we see that Definition 3.2 is equivalent to the existence of a function λ in U such that

$$f^*\tau = \lambda\tau. \tag{19}$$

This coincides with other definitions of a contact map on a contact manifold. In the basis $X_1, ..., X_m, T_1, ..., T_{N-m}$ the matrix of f_* has the form

$$\begin{pmatrix} Hf_* & \star \\ 0 & A \end{pmatrix} \tag{20}$$

where Hf_* is the map $f_*|HT_x \to HT_{f(x)}$ with $(f_*)_{ij} = X_j f_i$ and $\det A = \lambda$ with λ as above. We use the following notation:

$$|Hf_*(x)| = \max_{\xi \in HT_*, |\xi|=1} |Hf_*(x)\xi| \tag{21}$$

and

$$\ell[Hf_*(x)] = \min_{\xi \in HT_*, |\xi|=1} |Hf_*(x)\xi| \tag{22}$$

Definition 3.2 *A quasiregular map is a continuous (generalized) contact map $f : U \to G$ in $W^{1,N}_{loc}(U)$ which satisfies:*

$$|Hf_*(x)|^Q \leq K \det f_*(x) \tag{23}$$

and

$$\det f_*(x) \leq K\ell[Hf_*(x)]^Q \tag{24}$$

for some $K < \infty$ and a.a. $x \in U$.

For some constructions of quasiregular mappings and rigidity results in Carnot groups, see [11] and [5].

We Assume throughout this section that $f : U \to G$ is K-quasiregular. Notice that

$$|d_0 f_i(x)| \leq K^{2/Q} |d_0 f_j(x)| \tag{25}$$

for a.e. $x \in U$ and $i, j = 1, ..., m$

Next define $\hat{A}_x : HT_x \to HT_x$ by

$$\hat{A}_x(\xi) = \langle G_x(\xi), \xi \rangle^{(Q-2)} G_x(\xi), \tag{26}$$

where

$$G_x(\xi) = [\det f_*(x)]^{2/Q} [Hf_*(x)]^{-1} [Hf_*(x)]^{-1^T} \xi, \tag{27}$$

whenever $\det f_*(x) \neq 0$ (so that $Hf_*(x)$ is invertible), and $G_x(\xi) = \xi$ otherwise. From this we define $A_x : \Lambda^1 HT_x \to \Lambda^1 HT_x$ by $A_x = \gamma_1 \circ \hat{A}_x \circ \gamma^1$ where γ_1 is the linear map which sends basis elements of HT_x to those of $\Lambda^1 HT_x$ and $\gamma^1 = \gamma_1^{-1}$. Notice that since f is K-quasiregular, A_x is a Carathéodory function with $p = Q$ and $\nu = K$.

With A_x defined in this way we have the following result which appears as Corollary 3.20 in [5].

Theorem 3.3 *The first m components of a quasiregular mapping f are solutions to*

$$\delta A_x(d_0 u) = 0 \tag{28}$$

in U.

4. The Hardy-Littlewood inequality and John domains

Definition 4.1 *A bounded proper domain* $\Omega \subset G$ *is a* δ*-John domain,* $\delta > 0$*, if there exists a distinguished point* $x_0 \in \Omega$ *such that each* $x \in \Omega$ *can be joined with* x_0 *by a curve* $\gamma = \gamma : [0, l] \to \Omega$ *parametrized by arclength such that* $\gamma(0) = x, \gamma(l) = x_0$ *and* $d_c(\gamma(t), \Omega^c) \geq \delta t$*.*

We define Muckenhoupt weights. See [14].

Definition 4.2 *We write* $w \in A^q_M(\Omega), 1 < q < \infty, 1 \leq M < \infty$*, when* $w \geq 0$ *a.e. and*

$$\frac{1}{|B|}\int_B w \leq M \left(\frac{1}{|B|}\int_B \frac{1}{w^{1-q}}\right)^{1-q} \tag{29}$$

for all balls $B \subset \Omega$*.*

Muckenhoupt weights satisfy a reverse Holder inequality,

$$\int_B w^\beta \leq C|B|^{(1-\beta)/\beta}\int_B w \tag{30}$$

for all balls $B \subset \Omega$ and some $\beta > 1$.

We now state the main result of this paper.

Theorem 4.3 *Suppose that* $f = (f_1, ..., f_N)$ *is K-quasiregular in a* δ*-John domain with center* z_0*,* $\Omega \subset G$*,* $w \in A^q_M(\Omega)$ *and* $p > 0$*. There exists a constant* C*, independent of* f*, such that*

$$\int_\Omega |f_i - f_i(z_0)|^p w \leq C \int_\Omega |f_j|^p w \tag{31}$$

for all $i, j = 1, ..., m$*. Here* $C = C(\delta, p, N, Q, M, K)$*.*

John domains in homogeneous spaces satisfy a Boman chain condition, see [15], which allow the patching together of weak local L^p- estimates into global estimates. A proof of the following result appears in [4] in the Euclidean case and the proof extends here verbatim.

Theorem 4.4 *Let* $0 < p < \infty, \sigma > 1, w \in A^q_M(\Omega)$ *and let* u *and* v *be measurable functions defined in a* δ*-John domain* Ω*. Suppose there is a constant* A *such that*

$$\inf_{c\in\mathbb{R}} \int_B |u - c|^p w \leq A \int_{\sigma B} |v|^p w, \tag{32}$$

for all balls $\sigma B \subset \Omega$*. Then there is a constant* B*, independent of* u *and* v*, such that*

$$\inf_{c\in\mathbb{R}} \int_\Omega |u - c|^p w \leq B \int_\Omega |v|^p w. \tag{33}$$

Applying Theorem 4.4, we obtain the main result by proving the local result. The proof is similar to that in [17]. We first prove an unweighted inequality. Combining (15),(25) and (16),

$$\int_{2B} |f_i - (f_i)_{2B}|^\alpha \tag{34}$$

$$\le C_1 r \int_{2B} |d_0 f_i|^\alpha \tag{35}$$

$$\le C_4 r \int_{2B} |d_0 f_j|^\alpha \tag{36}$$

$$\le C_5 \int_{4B} |f_j|^\alpha. \tag{37}$$

Next we use the Holder inequalities (17) to get the local weighted result (32) with $\sigma = 8$ as follows.

$$\left(\int_B |f_i - (f_i)_{2B}|^p w\right)^{1/p} \tag{38}$$

$$\le \left(\int_B w^\beta\right)^{1/p\beta} \left(\int_B |f_i - (f_i)_{2B}|^{p\beta/(\beta-1)}\right)^{(\beta-1)/p\beta} \tag{39}$$

$$\le C_6 \left(\int_B w\right)^{1/p} |B|^{-1/\alpha} \left(\int_{2B} |f_i - (f_i)_{2B}|^\alpha\right)^{1/\alpha} \tag{40}$$

$$\le C_7 \left(\int_B w\right)^{1/p} |B|^{-1/\alpha} \left(\int_{4B} |f_j|^\alpha\right)^{1/\alpha} \tag{41}$$

$$\le C_8 \left(\int_B w\right)^{1/p} |B|^{q/p} \left(\int_{8B} |f_j|^{p/q}\right)^{q/p} \tag{42}$$

$$\le C_8 \left(\left(\int_{8B} w\right) |B|^{-q} \left(\int_{8B} w^{1/(1-q)}\right)^{-1}\right)^{1/p} \left(\int_{8B} |f_j|^p w\right)^{1/p} \tag{43}$$

$$\le C_9 \left(\int_{8B} |f_j|^p w\right)^{1/p}. \tag{44}$$

Because balls $B \subset G$ are John domains, we obtain the following corollary. We use the notations for the Hardy-Littlewood sharp maximal function and the sharp BMO norm in an open set U:

$$M_p^\sharp(u,\mu)(x) = \sup_{\substack{B\subset U \\ x\in B}} \mu(B)^{-1/p} \inf_{a\in\mathbb{R}} [\int_B |u-a|^p d\mu]^{1/p}, \tag{45}$$

and

$$\|u\|_{U,\mu}^{BMO} = \sup_{x\in U} M_1^\sharp(u,\mu)(x). \tag{46}$$

Here $d\mu = w dx$ for $w \in A_M^q(U)$.

Corollary 4.5 *Suppose that $f : U \to G$ is a qusiregular mapping. Then*

$$M_p^\sharp(f_i,\mu)(x) \le CM_p^\sharp(f_j,\mu)(x) \quad (47)$$

for $x \in U$ and

$$\|f_i\|_{U,\mu}^{BMO} \le C\|f_j\|_{U,\mu}^{BMO} \quad (48)$$

for $i, j = 1, ..., m$ where C is a constant independent of f.

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