VALIDITY OF THE ONE AND ONE-HALF LAYER QUASI-GEOSTROPHIC MODEL AND EFFECTIVE TOPOGRAPHY

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Abstract

We rigorously justify the so-called one and one-half layer quasi-geostrophic model from the two layer model as the ratio of the depth of the bottom layer over that of the top layer approaches infinity. The effective dynamics is given by the classical barotropic quasi-geostrophic dynamics for the bottom layer without topography, and the one layer quasi-geostrophic dynamics with the stream function of the bottom layer serving as an effective (possibly time dependent) topography for the the top layer. Such an one and one-half layer model is utilized in recent successful quantitative prediction of the Great Red Spot on Jupiter (see Turkington *et al* (2001)).

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1 INTRODUCTION

One of the simplest useful models in geophysical fluid dynamics that takes into consideration of rotation and stratification is the following **two layer model for quasi-geostrophic flows** (see Pedlosky (1979), Gill (1982) for instance)

$$\frac{\partial}{\partial t}(\Delta\psi_1 - F_1(\psi_1 - \psi_2) + \beta y) + J(\psi_1, \Delta\psi_1 - F_1(\psi_1 - \psi_2) + \beta y) = \mathcal{F}_1 \quad (1)$$

$$\frac{\partial}{\partial t}(\Delta\psi_2 - F_2(\psi_2 - \psi_1) + \beta y) + J(\psi_2, \Delta\psi_2 - F_2(\psi_2 - \psi_2) + \beta y) = \mathcal{F}_2 \quad (2)$$

where ψ_1 and ψ_2 are stream functions for the upper and lower layer of fluids respectively, $\mathcal{F}_j$ represents external forcing on the jth layer, $J(f_1, f_2) = \nabla^\perp f_1 \cdot \nabla f_2$ is the Jacobian,

$$F_j = \frac{f_0^2 L^2}{g(\frac{\Delta\rho}{\rho_0})D_j} \quad (3)$$

with

- $f = f_0 + \beta y$ being the Coriolis parameter,

- L represents typical horizontal length scale,

- g is the gravitation constant,

- D_j represents the depth of the jth layer,

- ρ_0 represents characteristic fluid density, and

- $\Delta\rho = \rho_2 - \rho_1 > 0$ represents the difference in density, and we consider a stably stratified case with lighter fluids (density ρ_1) sitting on top of heavier fluids (density ρ_2).

The boundary condition under consideration is the **channel geometry** where all unknowns are periodic in the zonal (longitude x) direction with period L and no-penetration / no flow boundary condition in the meridional (y) direction. More precisely, we have

$$\psi_j(x + L, y) = \psi_j(x, y), \quad (4)$$

$$\psi_j(x, 0) = \psi_j(x, h) = 0. \quad (5)$$

Such a set of boundary condition is more appealing than the usual doubly periodic boundary conditions. Physically the no-penetration in the latitudinal direction is closer to physics for flows on mid-latitude beta planes than the periodic boundary conditions, and the channel geometry (not the periodic geometry) allows us to derive conservation in time of the maximum modulus of the potential vorticity in the undamped/unforced environment(see for instance Chapter 1 of Majda and Wang 2004, and section 3 below).

Here we will consider the situation of a bottom layer much thicker than the top layer, i.e., $D_2 \gg D_1$. Physically this could happen in many situations, for instance, in the ocean with a relatively thin (mixing) layer sitting on a much thicker (inertial) layer. The effective dynamics at large ratio of depth (bottom over top) is the so-called one and one-half layer model (10-11). This simplified effective dynamics was central to the recent successful quantitative prediction of the Great Read Spot on the Jupiter by Turkington, Majda, Haven, and DiBattista (2001). A natural question then is if such an approximation is valid. The purpose of this short note is to present a rigorous mathematical justification of this one and one-half layer quasi-geostrophic model from the two layer model (1-2). More precisely, we demonstrate the following.

Theorem 1 *For sufficiently smooth initial data q_{j0}, $j = 1, 2$ and external forcing $\mathcal{F}_j$, $j = 1, 2$, the solutions ψ_j, $j = 1, 2$ of the two layer quasi-geostrophic equations (1-2) converge to the solutions ϕ_j, $j = 1, 2$ of the one and one-half layer quasi-geostrophic equations (10-11) as $\varepsilon = \frac{F_2}{F_1}$ approaches zero. Moreover, the convergence rate is ε, i.e., for any fixed time T, there exists a constant κ independent of ε such that*

$$\|\psi_j - \phi_j\|_{L^\infty(0,T;H^2)} \le \kappa\varepsilon, \quad j = 1, 2. \tag{6}$$

The rest of this short note is organized as follows. In section 2 we formally derive the one and one-half layer model from the two layer model as the depth of the bottom layer goes to infinity. The rigorous mathematical justification of such a limit, i.e., the proof of Theorem 1, is presented in section 3. We offer some concluding remarks and comments on the result obtained and some generalization in section 4.

2 FORMAL DERIVATION

Here we formally derive the one and one-half layer quasi-geostrophic model from the two layer quasi-geostrophic model assuming the bottom layer is

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much thicker than the top layer., i.e.,

$$\frac{D_1}{D_2} = \varepsilon \ll 1 \,. \tag{7}$$

Furthermore, we consider the case where the depth of the top layer and other parameters are fixed. Since the F_js are inversely proportional to the depth of the layer with the other parameters fixed (see (3)), we may quantify this assumption of the disparity of the depth of the two layers in the following relationship for the two layer model (1-2)

$$F_2 = \varepsilon F_1, \tag{8}$$

with F_1 been fixed.

We may then rewrite (2) as

$$\frac{\partial}{\partial t}(\Delta \psi_2 + \varepsilon F_1(\psi_1 - \psi_2)) + J(\psi_2, \Delta \psi_2 + \varepsilon F_1(\psi_1 - \psi_2)) + \beta \frac{\partial}{\partial x}\psi_2 = 0. \tag{9}$$

Formally setting $\varepsilon = 0$ in the above equation and holding other parameters including F_1 fixed, we arrive at the following **one and one-half layer quasi-geostrophic model** as effective dynamics in this deep bottom layer case

$$\frac{\partial}{\partial t}(\Delta \phi_1 - F_1\phi_1 + \beta y + F_1\phi_2) + J(\phi_1, \Delta \phi_1 - F_1\phi_1 + \beta y + F_1\phi_2) = \mathcal{F}_1 \tag{10}$$

$$\frac{\partial}{\partial t}(\Delta \phi_2 + \beta y) + J(\phi_2, \Delta \phi_2 + \beta y) = \mathcal{F}_2 \tag{11}$$

We notice that the effective dynamics of the bottom layer is independent of the top layer, and is governed by the barotropic quasi-geostrophic dynamics without topography.

On the other hand, denoting

$$h(\mathbf{x}, t) = F_1\phi_2(\mathbf{x}, t). \tag{12}$$

Then the effective dynamics of the top layer can be written as

$$\frac{\partial q}{\partial t} + J(\phi_1, q) = \mathcal{F}_1, \tag{13}$$

$$q = \Delta \phi_1 - F_1\phi_1 + \beta y + h.$$

This means a constant multiply of the bottom layer stream-function serves as a (possibly time dependent) **effective topography** for the top layer dynamics. This is precisely so in the case of steady state (stationary) bottom layer

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stream function $\phi_2(\mathbf{x})$. Such steady states are easily achieved by appropriate choice of the forcing term $\mathcal{F}_2(\mathbf{x})$ dictated by this given stream function. Moreover, many of these kind of stream-functions are nonlinearly stable (see Majda and Wang 2004, Chapter 4, among others).

The one and one-half layer model is intuitively correct. In the case of an extremely deep bottom layer, the inertial of the top layer is negligible when compared to that of the bottom layer and hence the bottom layer should not feel the top layer.

An alternative scaling of the layers is to assume a fixed bottom layer depth and a vanishing top layer depth. In this case the parameters are

$$F_1 = \frac{F_2}{\varepsilon}$$

with F_2 been fixed.

The effective dynamics derived by formally setting $\varepsilon = 0$ is then

$$\frac{\partial}{\partial t}(\psi_1 - \psi_2) + J(\psi_1, \psi_1 - \psi_2) = 0,$$

$$\frac{\partial}{\partial t}(\Delta\psi_2 - F_2(\psi_2 - \psi_1) + \beta y) + J(\psi_2, \Delta\psi_2 - F_2(\psi_2 - \psi_2) + \beta y) = \mathcal{F}_2.$$

However, we will not pursue this limit here.

3   MATHEMATICAL JUSTIFICATION

Here we rigorously justify the one and one-half layer model as the limit of the two layer model (1-2) as the ratio of the depth, i.e., ε, approaches zero.

3.1   A priori Estimates

As in almost all rigorous work on perturbation problems, we need to derive uniform estimates in terms of the perturbation variable ε (F_2). The natural way to derive such estimates is to utilize conserved quantities (in the unforced case) (see for instance Pedlosky 1987, Majda and Wang 2004 among others).

It turns out that the conserved quantity that is useful here is the maximum modulus of the potential vorticity in each layer in this channel geometry. We want to emphasize that the potential vorticities are conserved for our channel geometry but not necessarily conserved for the periodic geometry

(see for instance Majda and Wang 2004, Chapter 1, and the calculations below). Indeed, the two layer model (1-2) can be viewed as the potential vorticities $q_j = \Delta \psi_j + (-1)^j F_j(\psi_1 - \psi_2) + \beta y$ being transported along each particle trajectory if there were no external forcing ($\mathcal{F}_j = 0$), and thus the maximum modulus of the potential vorticities remain the same along each particle trajectory. Moreover, for the channel geometry, trajectories start within the channel never leave the channel which ensures the conservation of the modulus of the potential vorticity. In our forced situation, we may multiply the equations by q_j^{2k-1}, integrate over the channel and apply Hölder's inequality to obtain

$$\frac{d}{dt}(\frac{1}{2k}\|q_j\|_{L^{2k}}^{2k}) \leq \|\mathcal{F}_j\|_{L^{2k}}\|q_j\|_{L^{2k}}^{2k-1} \tag{14}$$

where the nonlinear term drops out since

$$
\begin{aligned}
\int_\Omega \nabla^\perp \psi_j \cdot \nabla q_j \, q_j^{2k-1} &= \frac{1}{2k}\int_\Omega \nabla^\perp \psi_j \cdot \nabla q_j^{2k} \\
&= \frac{1}{2k}\int_\Omega (-\frac{\partial \psi_j}{\partial y}\frac{\partial q_j^{2k}}{\partial x} + \frac{\partial \psi_j}{\partial x}\frac{\partial q_j^{2k}}{\partial y}) \\
&= \frac{1}{2k}\{-\int_{y=h} \psi_j(x,h)\frac{\partial q_j^{2k}}{\partial x}(x,h) + \int_{y=0} \psi_j(x,0)\frac{\partial q_j^{2k}}{\partial x}(x,0) \\
&\quad + \int_{x=L} \psi_j(L,y)\frac{\partial q_j^{2k}}{\partial y}(L,y) - \int_{x=0} \psi_j(0,y)\frac{\partial q_j^{2k}}{\partial y}(0,y)\} \\
&= 0 \tag{15}
\end{aligned}
$$

where we performed integration by parts and utilized the channel boundary condition (5) in an essential way (the q_j's are not periodic in y), and the periodicity (with period L) of ψ_j and q_j in x.

Inequality (14) implies

$$\|q_j\|_{L^\infty(0,T;L^{2k})} \leq \|q_{j0}\|_{L^{2k}} + T|\Omega|^{1/2k}\|\mathcal{F}_j\|_{L^\infty(0,T;L^\infty)} \tag{16}$$

which further implies

$$
\begin{aligned}
\|q_j\|_{L^\infty(0,T;L^\infty)} &= \lim_{k\to\infty}\|q_j\|_{L^\infty(0,T;L^{2k})} \\
&\leq \|q_{j0}\|_{L^\infty} + T\|\mathcal{F}_j\|_{L^\infty(0,T;L^\infty)}. \tag{17}
\end{aligned}
$$

Notice that this estimate is not uniform in ε estimate for ψ_j (or q_j). In order to get uniform estimates, we first derive uniform estimate for the difference of the stream function by forming the difference of the two inequalities

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in (17). We have

$$
\begin{aligned}
& \|\Delta(\psi_1 - \psi_2) - (F_1 + F_2)(\psi_1 - \psi_2)\|_{L^\infty(0,T;L^\infty)} \\
= \ & \|\Delta\psi_1 - F_1(\psi_1 - \psi_2) + \beta y - (\Delta\psi_2 - F_2(\psi_2 - \psi_1) + \beta y)\|_{L^\infty(0,T;L^\infty)} \\
\leq \ & \|q_1\|_{L^\infty(0,T;L^\infty)} + \|q_2\|_{L^\infty(0,T;L^\infty)} \\
\leq \ & \kappa_1 \hspace{8cm} (18)
\end{aligned}
$$

where we utilized simple triangular inequality and (17). Here and elsewhere, the κ_js denote generic constants independent of ε.

The previous inequality implies, since $F_1 + F_2 \geq 0$,

$$
\|\Delta(\psi_1 - \psi_2)\|_{L^\infty(0,T;L^\infty)} \leq \kappa_2 \tag{19}
$$

Combining this with the uniform estimates on the potential vorticity (17), we have

$$
\|\Delta\psi_j\|_{L^\infty(0,T;L^\infty)} \leq \kappa_3, \quad j = 1, 2. \tag{20}
$$

The global in time wellposedness of the two layer model follows from these two estimates together with the usual analysis for classical two dimensional inviscid flows which can be found in Majda and Bertozzi (2001) among others. An alternative way to understand this is that, since the vorticities and potential vorticities remain bounded for all time, the Beale-Kato-Majda type criterion tells us that classical solution must exist for all time.

We also claim that the same kind of estimates hold for the limit one and one-half layer model (10-11). Moreover, since the bottom layer of the one and one-half layer model satisfies the barotropic quasi-geostrophic equation without topography, we can copy regularity works on two dimensional Euler equations for incompressible perfect fluids and derive higher order estimates on the solutions (see Majda and Bertozzi 2001, section 4.4, Temam 1975 among others). This in turn implies higher order regularity on the top layer since the stream function of the bottom layer serves as scaled effective topography for the top layer. In particular we have

$$
\|\phi_j\|_{L^\infty(0,T;H^s)} \leq \kappa_4, \ j = 1, 2 \tag{21}
$$

for some sufficiently high index s.

3.2 Convergence

We are now ready to prove the convergence, i.e., Theorem 1.

We introduce the difference between the stream functions

$$\tilde{\psi}_j = \psi_j - \phi_j, \ j = 1, 2. \tag{22}$$

We first look into the equation satisfied by the difference of the bottom layer stream functions. This is expected to be relatively easier due to the simplified dynamics for the bottom layer in the one and one-half layer model (10-11). We then have, utilizing the two layer model (1-2) and the one and one-half layer model (10-11),

$$\frac{\partial}{\partial t}\Delta\tilde{\psi}_2 - F_2\frac{\partial}{\partial t}(\psi_2 - \psi_1) + J(\psi_2, \Delta\tilde{\psi}_2) + J(\tilde{\psi}_2, \Delta\phi_2) - F_2 J(\psi_2, \psi_2 - \psi_1) + \beta\frac{\partial\tilde{\psi}_2}{\partial x} = 0 \tag{23}$$

Next we multiply this equation by $\Delta\tilde{\psi}_2$ and integrate over the channel. Notice that the nonlinear terms can be estimated as, after performing elementary integration by parts, utilizing (20), (21), as well as Sobolev imbedding,

$$
\begin{aligned}
\int \frac{\partial\tilde{\psi}_2}{\partial x}\Delta\tilde{\psi}_2 &= -\frac{1}{2}\int \frac{\partial}{\partial x}|\nabla\tilde{\psi}_2|^2 \\
&= 0, \tag{24} \\
\int J(\psi_2, \Delta\tilde{\psi}_2)\Delta\tilde{\psi}_2 &= \int \nabla^\perp\psi_2 \cdot \nabla\Delta\tilde{\psi}_2\Delta\tilde{\psi}_2 \\
&= \frac{1}{2}\int \nabla^\perp\psi_2 \cdot \nabla(\Delta\tilde{\psi}_2)^2 \\
&= 0, \tag{25} \\
\left|\int J(\psi_2, \psi_2 - \psi_1)\Delta\tilde{\psi}_2\right| &= \left|\int \nabla^\perp\psi_2 \cdot \nabla(\psi_2 - \psi_1)\Delta\tilde{\psi}_2\right| \\
&\leq |\Delta\tilde{\psi}_2|_{L^2}|\nabla\psi_2|_{L^2}|\nabla(\psi_2 - \psi_1)|_{L^\infty} \\
&\leq \kappa_5|\Delta\tilde{\psi}_2|_{L^2}, \tag{26} \\
\left|\int J(\tilde{\psi}_2, \Delta\phi_2)\Delta\tilde{\psi}_2\right| &= \left|\int \nabla^\perp\tilde{\psi}_2 \cdot \nabla\Delta\phi_2\Delta\tilde{\psi}_2\right| \\
&\leq |\nabla\tilde{\psi}_2|_{L^4}|\nabla\Delta\phi_2|_{L^4}|\Delta\tilde{\psi}_2|_{L^2} \\
&\leq \kappa_6|\Delta\tilde{\psi}_2|_{L^2}^2. \tag{27}
\end{aligned}
$$

Next, we need to estimate the time derivative of the difference of the stream functions of the top and bottom layer as suggested by (23). For this purpose, we take the difference of equations (1) and (2) and we obtain

$$\frac{\partial}{\partial t}((\Delta - (F_1 + F_2))(\psi_1 - \psi_2)) = -J(\psi_1, q_1) + J(\psi_2, q_2) + \mathcal{F}_1 - \mathcal{F}_2 \tag{28}$$

which can be re-written as

$$\frac{\partial}{\partial t}(\psi_1 - \psi_2) = (\Delta - (F_1 + F_2))^{-1}(-J(\psi_1, q_1) + J(\psi_2, q_2) + \mathcal{F}_1 - \mathcal{F}_2) \quad (29)$$

Thus, we have

$$
\begin{aligned}
|\frac{\partial}{\partial t}(\psi_2 - \psi_1)|_{H^1} &= |(\Delta - (F_1 + F_2))^{-1}(-J(\psi_1, q_1) + J(\psi_2, q_2) + \mathcal{F}_1 - \mathcal{F}_2)|_{H^1} \\
&\leq \kappa_7 |J(\psi_1, q_1) - J(\psi_2, q_2) - \mathcal{F}_1 + \mathcal{F}_2|_{H^{-1}} \\
&\leq \kappa_8 \{|\nabla \psi_1|_{L^2}|q_1|_{L^\infty} + |\nabla \psi_2|_{L^2}|q_2|_{L^\infty} + |\mathcal{F}_1 - \mathcal{F}_2|_{H^{-1}}\} \\
&\leq \kappa_9.
\end{aligned}
\quad (30)
$$

where we have used (17) and the following simple inequality

$$|\int J(f, g)h| = |\int J(f, h)g| \leq |\nabla f|_{L^2}|g|_{L^\infty}|\nabla h|_{L^2} \quad (31)$$

which implies that

$$|J(f, g)|_{H^{-1}} \leq |\nabla f|_{L^2}|g|_{L^\infty}. \quad (32)$$

Combining (24-27), (30) and utilizing Cauchy-Schwarz we have

$$\frac{1}{2}\frac{d}{dt}|\Delta\tilde{\psi}_2|_{L^2}^2 \leq \kappa_{10}|\Delta\tilde{\psi}_2|_{L^2}^2 + \kappa_{11}\varepsilon^2. \quad (33)$$

which further implies

$$\|\psi_2 - \phi_2\|_{L^\infty(0,T;H^2)} \leq \kappa_{12}\varepsilon. \quad (34)$$

This is the second estimate in Theorem 1.

As a byproduct, we can estimate the time derivative of the difference of the bottom layer stream functions. This is needed in proving the convergence of the top layer as we shall see below. For this purpose we go back to (23) and notice

$$\frac{\partial}{\partial t}\tilde{\psi}_2 = \Delta^{-1}(F_2\frac{\partial}{\partial t}(\psi_2 - \psi_1) - J(\psi_2, \Delta\tilde{\psi}_2) - J(\tilde{\psi}_2, \Delta\phi_2) + F_2 J(\psi_2, \psi_2 - \psi_1) - \beta\frac{\partial\tilde{\psi}_2}{\partial x})$$
$$(35)$$

Thus, we have, after applying elementary integration by parts and utilizing estimates (20), (21), (30), (32) and (34),

$$|\frac{\partial}{\partial t}\tilde{\psi}_2|_{H^1} \leq F_2|\frac{\partial}{\partial t}(\psi_2 - \psi_1)|_{H^{-1}} + |J(\psi_2, \Delta\tilde{\psi}_2)|_{H^{-1}}$$

$$+|J(\tilde{\psi}_2, \Delta\phi_2)|_{H^{-1}} + F_2|J(\psi_2, \psi_2 - \psi_1)|_{H^{-1}} + \beta|\frac{\partial\tilde{\psi}_2}{\partial x}|_{H^{-1}})$$

$$\leq \kappa_{13}\varepsilon + \kappa_{14}|\nabla\psi_2|_{L^\infty}|\Delta\tilde{\psi}_2|_{L^2}$$

$$\leq \kappa_{15}\varepsilon . \tag{36}$$

We now proceed to prove the convergence of the top layer.

For this purpose we take the difference of the equations for the top layer in the two layer model (1) and that of the one and one-half layer model (10), we deduce

$$\frac{\partial}{\partial t}(\Delta\tilde{\psi}_1 - F_1\tilde{\psi}_1) = -F_1\frac{\partial}{\partial t}\tilde{\psi}_2 - J(\psi_1, (\Delta - F_1)\tilde{\psi}_1) - J(\tilde{\psi}_1, (\Delta - F_1)\phi_1)$$

$$-J(\tilde{\psi}_1, F_1\phi_2) - J(\psi_1, F_1\tilde{\psi}_2) - \beta\frac{\partial\tilde{\psi}_1}{\partial x}. \tag{37}$$

The estimates are very similar as for the bottom layer as given in (24-32). Indeed, we have, after multiplying (37) by $(\Delta - F_1)\tilde{\psi}_1$ and integrating over the channel,

$$\frac{1}{2}\frac{d}{dt}|(\Delta - F_1)\tilde{\psi}_1|^2_{L^2} \leq \kappa_{16}|(\Delta - F_1)\tilde{\psi}_1|^2_{L^2} + \kappa_{17}\varepsilon^2, \tag{38}$$

where we have utilized the following inequalities which in turn relies on elementary integration by parts and estimates (20), (21), (34), (36),

$$|\int \frac{\partial}{\partial t}\tilde{\psi}_2(\Delta - F_1)\tilde{\psi}_1| \leq \kappa_{15}\varepsilon|(\Delta - F_1)\tilde{\psi}_1|_{L^2}, \tag{39}$$

$$\int J(\psi_1, (\Delta - F_1)\tilde{\psi}_1)(\Delta - F_1)\tilde{\psi}_1 = 0 \tag{40}$$

$$|\int J(\tilde{\psi}_1, (\Delta - F_1)\phi_1)(\Delta - F_1)\tilde{\psi}_1| \leq \kappa_{18}|(\Delta - F_1)\tilde{\psi}_1|^2_{L^2}, \tag{41}$$

$$|\int J(\tilde{\psi}_1, F_1\phi_2)(\Delta - F_1)\tilde{\psi}_1| \leq \kappa_{19}|(\Delta - F_1)\tilde{\psi}_1|^2_{L^2}, \tag{42}$$

$$|\int J(\psi_1, F_1\tilde{\psi}_2)(\Delta - F_1)\tilde{\psi}_1| \leq \kappa_{20}\varepsilon|(\Delta - F_1)\tilde{\psi}_1|_{L^2}, \tag{43}$$

$$\int \frac{\partial\tilde{\psi}_1}{\partial x}(\Delta - F_1)\tilde{\psi}_1 = 0. \tag{44}$$

It then follows from (38) the convergence of the top layer, namely,

$$\|\psi_1 - \phi_1\|_{L^\infty(0,T;H^2)} \leq \kappa_{21}\varepsilon. \tag{45}$$

This completes the proof of the theorem.

4 REMARKS AND COMMENTS

We have established the validity of the one and one-half layer quasi-geostrophic model heuristically and rigorously from the two layer model. The convergence rate is ε which is the ratio of the depths of the top thin layer over that of the thick bottom layer. The analysis is somewhat straightforward since as we observe, the perturbation problem is a regular perturbation one (no initial, boundary or internal layer/singularity emerge as the small parameter approaches zero). In fact, we can derive systematic asymptotic expansion of the solutions of the two layer model in terms of the small parameter ε. The systematic expansion demonstrates that the convergence rate of ε is optimal. We will not pursue the details here.

Here we have discussed the situation of forced inviscid flows. Most physical problems involve both forcing and damping/dissipation. It seems that the analysis presented here may be generalized to certain special types of dissipation, for instance, for systems of the form

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = -d_1(\Delta\psi_1 - F_1(\psi_1 - \psi_2)) + \mathcal{F}_1, \qquad (46)$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = -d_2(\Delta\psi_2 - F_2(\psi_2 - \psi_1)) + \mathcal{F}_2 \qquad (47)$$

where the non-negative coefficients d_j's are the analogies for Ekman damping coefficient. Such kind of non-symmetric dissipations have been proposed by oceanographers (see for instance Holloway (1992)). The interested reader may work out the details. Nevertheless, it is still not clear how to handle the problem with classical (symmetric) Ekman damping, eddy viscosity etc.

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