

# CELESTIAL INTEGRATION, STRINGY INVARIANTS, AND CHERN-SCHWARTZ-MACPHERSON CLASSES

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ABSTRACT. We introduce a formal integral on the system of varieties mapping properly and birationally to a given one, with value in an associated Chow group. Applications include comparisons of Chern numbers of birational varieties, new birational invariants, ‘stringy’ Chern classes, and a ‘celestial’ zeta function specializing to the topological zeta function.

In its simplest manifestation, the integral gives a new expression for Chern-Schwartz-MacPherson classes of possibly singular varieties, placing them into a context in which a ‘change of variable’ formula holds.

The formalism has points of contact with motivic integration.

## 1. INTRODUCTION

**1.1.** In this note I review the notion of *celestial integration*, and sketch a few applications; for proofs and further details, the reader is addressed to [Alu05]. I am very grateful to Jean-Paul and Cidinha for organizing the *VIIIème Rencontre Internationale de São Carlos sur les singularités réelles et complexes au CIRM*, relocating for the occasion the idyllic surroundings of *São Carlos, Brasil* to the idyllic surroundings of *Luminy, France*. Perfect weather, exceptionally interesting talks, and spirited conversations made the conference a complete success. What follows was the subject of my lecture at the São Carlos/Luminy meeting, and preserves (for better or worse) the informal nature of a seminar talk.

I thank the Max-Planck-Institut für Mathematik in Bonn, where much of this material was conceived and where this note was written.

**1.2.** Summary: to a variety  $X$  I will associate a large group  $A_*\mathcal{C}_X$  (containing the Chow group  $A_*X$  of  $X$ ); for certain data  $\mathcal{D}$ ,  $\mathcal{S}$  (arising, for example, from a divisor  $D$  and a constructible subset  $S$  of  $X$ ) I will define a distinguished element

$$\int_S \mathbb{1}(\mathcal{D}) d\mathbf{c}_X \in A_*\mathcal{C}_X \quad .$$

These are the *celestial integrals* in the title; they are not defined as integrals, but satisfy formal properties justifying the terminology. The *celestial* qualifier is meant to evoke the fact that the *modification systems* on which this operation is defined are close relatives of Hironaka’s *voûte étoilée*.

Applications of this construction:

- Comparison of Chern classes of birational varieties;
- Birational invariants;
- ‘Celestial’ zeta functions;
- Invariants of singular varieties (‘*Stringy Chern classes*’);
- Relations with the theory of Chern-Schwartz-MacPherson classes.

**1.3. Digression: motivic integration.** This is technically not necessary for the rest of the talk, but useful nonetheless as ‘inspiration’ for the main construction.

The *Grothendieck group of varieties* is the free abelian group on symbols  $[X]$ , where  $X$  is a complex algebraic variety up to isomorphism, modulo relations

$$[X] = [Y] + [X \setminus Y]$$

for each closed subvariety  $Y \subset X$ . This group may be made into a ring by setting  $[X] \cdot [Y] = [X \times Y]$ .

*Example 1.1.* The class of a point is the identity for this multiplication. The class  $[\mathbb{A}^1]$  is denoted  $\mathbb{L}$ ; thus

$$[\mathbb{P}^n] = [\mathbb{A}^0] + [\mathbb{A}^1] + \cdots + [\mathbb{A}^n] = \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1} \quad .$$

As Eduard Looijenga writes ([Loo02], p. 269), this ring—or rather its localization at  $\mathbb{L}$ —is *interesting, big, and hard to grasp*. In practice, it is necessary to further tweak this notion, by a suitable completion with respect to a dimension filtration; I will glibly ignore such important ‘details’.

Mapping  $X$  to its class in this ring gives a *universal Euler characteristic*: anything satisfying the basic relations—e.g., topological Euler characteristic, Hodge polynomials and structures, . . . , must factor through this map. This is motivation to ‘compute’  $[X]$  for given  $X$ .

Through *motivic integration*, one can determine an element

$$\int_{\mathcal{S}} \mathbb{L}^{-\text{ord } D} d\mu$$

in the completed Grothendieck ring of varieties, from the information of a divisor  $D$  and a constructible subset  $\mathcal{S}$  of the arc space  $\mathcal{X}$  of  $X$ . If  $X$  is nonsingular, then choosing  $D = 0$ ,  $\mathcal{S} = \mathcal{X}$  gives  $\int_{\mathcal{X}} \mathbb{L}^0 d\mu = [X]$ .

Motivic integration was defined and developed by Maxim Kontsevich, Jan Denef, and François Loeser, and it is of course much deeper than this brief summary can begin to suggest. There are many good surveys of this material, for example: [Cra04], [DL01], [Loo02], [Vey05]. For an explanation of what makes motivic integration *motivic*, see the appendix in [Cra04].

The definition of the integral  $\int_{\mathcal{S}} \mathbb{L}^{-\text{ord } D} d\mu$  relies on the study of the *arc spaces* of a variety;  $d\mu$  is a measure on this space, with value in the completed Grothendieck ring; the integral is an honest integral with respect to this measure, and as such it satisfies a *change of variable* formula: for example, if  $\pi : Y \rightarrow X$  is proper and birational, then

$$\int_{\mathcal{X}} \mathbb{L}^{-\text{ord } D} d\mu = \int_{\mathcal{Y}} \mathbb{L}^{-\text{ord}(\pi^{-1}D + K_{\pi})} d\mu \quad ,$$

where  $K_{\pi}$  denotes the relative canonical sheaf.

This formula is at the root of spectacular applications of motivic integration. For example, suppose  $X, Y$  are *birational* nonsingular, complete Calabi-Yau varieties;

resolve a birational morphism between them:

$$\begin{array}{ccc}
 & V & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 X & \text{-----} & Y
 \end{array}$$

with  $V$  nonsingular and  $\pi_1, \pi_2$  proper and birational. Then the Calabi-Yau condition implies  $K_{\pi_1} = K_{\pi_2}$ ; denoting this by  $K$  gives

$$[X] = \int_{\mathcal{X}} \mathbb{L}^0 d\mu = \int_{\mathcal{V}} \mathbb{L}^{-\text{ord } K} d\mu = \int_{\mathcal{Y}} \mathbb{L}^0 d\mu = [Y] \quad .$$

Hence: such varieties must have the same topological Euler characteristic, Betti numbers, Hodge polynomials, etc.

**1.4. Motivic integration** only serves as *motivation* for the rest of this lecture, or maybe more correctly as a motivating analogy. The basic relation in the Grothendieck ring holds (in a suitable sense) for *Chern-Schwartz-MacPherson* classes; ‘hence’ there should be a ‘motivic’ theory of such classes: it should be possible to deal with the classes within the framework of an integration theory, satisfying a suitable change-of-variable formula; one should be able to play tricks such as the application sketched above at the level of Chern classes.

This is the guiding theme in what follows.

## 2. MODIFICATION SYSTEMS

**2.1.** The task is to define an ‘integral’ carrying information about Chern classes. Taking at heart the lesson learned in motivic integration, we should start by defining an appropriate context in which this integral may take its value.

Let  $X$  be a variety over an algebraically closed field of characteristic zero (the precise requirement is that embedded resolution à la Hironaka should work).

**Definition 2.1.** I will denote by  $\mathcal{C}_X$  the category of proper birational maps

$$\begin{array}{c}
 V_\pi \\
 \downarrow \pi \\
 X
 \end{array}$$

with morphisms given in the obvious way by commutative triangles

$$\begin{array}{ccc}
 V_{\pi_1} & \xrightarrow{\alpha} & V_{\pi_2} \\
 \pi_1 \searrow & & \swarrow \pi_2 \\
 & X &
 \end{array}$$

with  $\alpha$  proper and birational.

Proper birational maps are often called *modifications*, and the natural way to think of  $\mathcal{C}_X$  is as an inverse system, so it seems appropriate to call this category the *modification system* of  $X$ . Also, it is useful to take this notion up to the following equivalence relation: say that  $\mathcal{C}_X$  and  $\mathcal{C}_Y$  are equivalent if there exist objects in  $\mathcal{C}_X$  and  $\mathcal{C}_Y$  with

a common source. For example, if  $X$  and  $Y$  are birational and complete then their modification systems are equivalent in this sense.

**2.2.** I will (usually) denote by  $V_\pi$  the source of the object  $\pi$  of  $\mathcal{C}_X$ . It is hard to resist the temptation to think of the object  $\pi$  really in terms of its corresponding  $V_\pi$ , and of  $\mathcal{C}_X$  as a system of varieties birational to  $X$ .

Morally I would like to take the inverse limit of this system, and define ordinary data such as divisors, Chow group, etc. for the resulting *provariety*. In practice, it is more straightforward to simply define these data as appropriate limits of the corresponding data on the individual  $V_\pi$ 's. For example, denote by  $A_*V_\pi$  the Chow group of  $V_\pi$ , with rational coefficients; then

$$\mathcal{A}_X := \{A_*V_\pi \mid \pi \in \text{Ob}(\mathcal{C}_X)\}$$

is an inverse system of abelian groups under proper push-forward.

**Definition 2.2.** The *Chow group* of  $\mathcal{C}_X$  is the inverse limit of this system:

$$A_*\mathcal{C}_X := \varprojlim \mathcal{A}_X \quad .$$

Thus, an element  $a \in A_*\mathcal{C}_X$  consists of the data of a class  $(a)_{\text{id}}$  in the Chow group of  $X$  and of compatible lifts  $(a)_\pi$  for all  $\pi \in \text{Ob}(\mathcal{C}_X)$ . I call  $(a)_\pi$  the  $\pi$ -*manifestation* of  $a$ .

Note that any class  $\alpha \in A_*X$  determines a ‘silly’ class  $a \in A_*\mathcal{C}_X$ : just set  $(a)_\pi := \pi^*\alpha$ . One intriguing (to me, at least) consequence of the construction given in this paper is that certain classes on  $X$  have *other*, more interesting, lifts to  $A_*\mathcal{C}_X$ . These lifts call for rational coefficients, hence the need for rational coefficients in the definition of  $A_*\mathcal{C}_X$ .

Equivalent modification systems have isomorphic Chow groups.

**2.3.** Other standard notions may be defined similarly. Divisors and constructible sets of sources  $V_\pi$  are organized by *direct* systems, under pull-backs; the corresponding notions for a modification system are defined as direct limits of these systems.

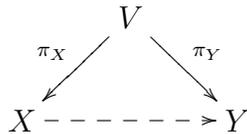
For example, a divisor  $\mathcal{D}$  of  $\mathcal{C}_X$  is represented by a pair  $(\pi, D_\pi)$  with  $D_\pi$  a divisor of  $V_\pi$ , and where pairs  $(\pi, D_\pi), (\pi \circ \alpha, D_{\pi \circ \alpha})$  are identified whenever  $\alpha : V_{\pi \circ \alpha} \rightarrow V_\pi$  is a proper birational map and  $D_{\pi \circ \alpha} = \alpha^{-1}(D_\pi)$ :

$$V_{\pi \circ \alpha} \xrightarrow{\alpha} V_\pi \xrightarrow{\pi} X \quad .$$

An obvious way to get a divisor  $\mathcal{D}$  is by pulling back a divisor of  $X$  through the whole system; but note that there are *many other* divisors: for example, every subscheme  $S$  of  $X$  determines a divisor of  $\mathcal{C}_X$  (represented by the exceptional divisor in the blow-up of  $X$  along  $S$ ). As a bonus, equivalent modification systems have the same divisors, while birational varieties don't.

**2.4.** The story is entirely analogous for *constructible subsets* of a modification system. The ‘obvious’ such object is determined by a constructible (for example, closed) subset of  $X$ , by taking inverse images through the system. While this is our main example, one should keep in mind that the notion is considerably more general.

Of course, equivalent systems have the same constructible subsets. For example, if  $V$  maps properly and birationally to both  $X$  and  $Y$ :



then  $(\pi_X, V)$  determines the same subset of  $\mathcal{C}_X$  as  $(\text{id}, X)$  and the same subset of  $\mathcal{C}_Y$  as  $(\text{id}, Y)$ ; abusing language I may denote this object by  $\mathcal{C}_X$  or  $\mathcal{C}_Y$  according to the context, but the reader should keep in mind that these constructible subsets of different systems may be identified.

Details about all these notions, and natural definitions (such as sums of divisors, or unions of constructible subsets) are left to the interested reader, and may be found in [Alu05].

### 3. CELESTIAL INTEGRALS

**3.1.** The main result of this note is that for a variety  $X$ , a divisor  $\mathcal{D}$  of  $\mathcal{C}_X$ , and a constructible subset  $\mathcal{S}$ , there is an element (the ‘celestial integral’ of  $\mathcal{D}$  over  $\mathcal{S}$ , in  $\mathcal{C}_X$ )

$$\int_{\mathcal{S}} \mathbb{1}(\mathcal{D}) d\mathbf{c}_X$$

of the Chow group  $A_*\mathcal{C}_X$ , satisfying interesting properties.

The actual definition of this element is uninspiring; I’ll give it at the end of the paper for the sake of completeness. The properties satisfied by this ‘integral’ are more important, so they get the honor of prime time.

Of course the notion is additive with respect to disjoint unions of constructible subsets, as should be expected from an integral. What makes the notion interesting is that it computes interesting objects for suitable choices of the input data, and that it satisfies a change-of-variable formula (again, as should be expected from an integral!).

More explicitly:

**Theorem 3.1.** (1) (Normalization) *Assume  $X$  is nonsingular. If  $\mathcal{S}$  is represented by  $(\text{id}, S)$  with  $S \subset X$  a nonsingular subvariety, then*

$$\left( \int_{\mathcal{S}} \mathbb{1}(0) d\mathbf{c}_X \right)_{\text{id}} = c(TS) \cap [S] \quad ,$$

*the total homology Chern class of  $S$ , viewed as an element of  $A_*X$ .*

(2) (Change-of-variables) *If  $\rho : Y \rightarrow X$  is proper and birational, then*

$$\int_{\mathcal{S}} \mathbb{1}(\mathcal{D}) d\mathbf{c}_X = \int_{\mathcal{S}} \mathbb{1}(\mathcal{D} + K_\rho) d\mathbf{c}_Y \quad ,$$

*where  $K_\rho$  denotes the relative canonical divisor of  $\rho$ .*

The normalization property (1) is self-explanatory; it will serve as a point of depart for extensions to possibly *singular* subsets  $S$  of a nonsingular variety, in §§5 and 6. By contrast, (2) deserves an immediate clarification.

**3.2.** First of all, note that under the given hypotheses we have that  $\mathcal{C}_X$  and  $\mathcal{C}_Y$  are equivalent systems; thus we may indeed treat  $\mathcal{S}$  and  $\mathcal{D}$  as data belonging to either.

Secondly, I have to clarify what I mean by *the relative canonical divisor*  $K_\rho$  of  $\rho : Y \rightarrow X$ . It in fact turns out that there are more than one sensible such notions, according to the context. In the simplest case, when  $X$  and  $Y$  are nonsingular,  $K_\rho$  is the divisor corresponding to the determinant of the differential  $d\rho : TY \rightarrow \rho^*TX$ . In the general case there is a choice as to what is the correct generalization of  $TX$ ; I'll come back to this point in §5.2.

Further, in general the scheme corresponding to the vanishing of the determinant may not be locally principal. This is not a problem in our context, however, since every subscheme of every variety in the modification system  $\mathcal{C}_X$  determines a divisor of the system, as observed in §2.3.

#### 4. SKETCH OF APPLICATIONS

**4.1. Invariance of Chern classes.** Exactly as in the case of motivic integration, the change-of-variable formula yields an invariance statement for celestial integration across birational morphisms preserving the canonical class.

Two varieties  $X, Y$  (nonsingular, for simplicity) are *K-equivalent* if their modification systems are equivalent, and the canonical divisors  $K_X, K_Y$  agree in the system(s): that is, if the pull-backs of  $K_X, K_Y$  to a common source agree:

$$\begin{array}{ccc} & V & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & \text{-----} & Y \\ \pi_X^* K_X & = & \pi_Y^* K_Y \end{array} .$$

In this situation, letting  $K = K_{\pi_X} = K_{\pi_Y}$  and applying the change-of-variable formula (2) gives:

$$\int_{\mathcal{S}} \mathbb{1}(\mathcal{D}) d\mathbf{c}_X = \int_{\mathcal{S}} \mathbb{1}(\mathcal{D} + K) d\mathbf{c}_V = \int_{\mathcal{S}} \mathbb{1}(\mathcal{D}) d\mathbf{c}_Y .$$

Therefore:

**Theorem 4.1.** *Celestial integrals on K-equivalent varieties agree as elements of the (common) Chow group of the corresponding modification systems.*

**4.2.** For example, applying this observation with  $\mathcal{D} = 0$ ,  $\mathcal{S} = \mathcal{C}_X (= \mathcal{C}_Y \text{ on } Y)$  and using (1) from §3 shows that  $c(TX) \cap [X]$  and  $c(TY) \cap [Y]$  are manifestations (in  $A_*X, A_*Y$ , respectively) of *the same* class in the Chow group of the modification system.

This recovers the fact (known, for example, through motivic integration) that the Euler characteristics of *K-equivalent* varieties must agree. More generally, it shows (via a simple application of the projection formula) that all numbers

$$c_1^i \cdot c_{n-i}$$

with  $n = \dim X = \dim Y$  must agree for *K-equivalent* varieties.

Incidentally, these numbers must therefore be invariant through classical flops, and hence (as shown by Burt Totaro, [Tot00]) they must factor through the complex

elliptic genus. It is a pleasant exercise to verify this fact directly: these numbers can be assembled into a genus (which I would like to call the *cuspidal* genus, for reasons which will likely be apparent to many readers), corresponding to the *characteristic*  $e^{xT}(1+xU)$ ; it is straightforward to check directly that the cuspidal genus factors through the complex elliptic genus.

The invariance of Chern numbers mentioned above is of course only a very particular case of similar results accessible through celestial integration. Every choice of a divisor and a constructible subset yields an analogous (but, unfortunately, usually much less transparent) invariance statement.

**4.3. Birational invariants.** As another application, we can extract new birational invariants from the integral. For example, let

$$\mathrm{dCan}(X) := \left\{ \deg \int_{\mathcal{C}_X} \mathbb{1}(\mathcal{K}) d\mathbf{c}_X \mid \mathcal{K} \text{ effective canonical divisor of } X \right\} \subset \mathbb{Z} \quad .$$

**Theorem 4.2.** *If  $X, Y$  are birational complete varieties, then  $\mathrm{dCan}(X) = \mathrm{dCan}(Y)$ .*

For example: if  $X$  is birational to a Calabi-Yau variety  $Y$ , then  $\mathrm{dCan}(X) = \{\chi(Y)\}$ .

An interesting question is whether an analogous (and nontrivial) invariant can be defined for varieties without effective canonical divisors. A few natural candidates for such invariants, involving negative representatives, must be ruled out at the moment because of a sticky technical obstacle to the definition of celestial integrals for noneffective divisors (see §7).

**4.4. Zeta functions.** In analogy with motivic integration, zeta functions can be concocted from motivic integrals. For example, for a divisor  $D$  of  $X$  (say defined by  $f = 0$ ) set

$$Z(D, m) := \int_{\mathcal{C}_X} \mathbb{1}(m\mathcal{D}) d\mathbf{c}_X \quad ,$$

a series in the variable  $m$ , with coefficients in  $\mathcal{A}_*\mathcal{C}_X$ ; here  $\mathcal{D}$  is the divisor in  $\mathcal{C}_X$  determined by  $D$ . Then

**Theorem 4.3.** *The degree of  $Z(D, m)$  equals the topological zeta function of  $f$ .*

This connection makes it possible to formulate analogs of the *monodromy conjecture* (see for example [Vey05], §6.8) for celestial zeta functions. I hope that the celestial viewpoint will add something to the circle of ideas surrounding zeta functions. For example, conceivably the relationship between celestial integration and the theory of Chern-Schwartz-MacPherson classes (§6) may give a tool to compute local contributions to the zeta function of a hypersurface in terms of the Segre class of its singularity subscheme.

## 5. STRINGY INVARIANTS

**5.1.** If  $X$  has sufficiently mild singularities, there is a notion of *stringy Euler characteristic* of  $X$ , introduced by Batyrev. For example, in the particular case in which  $X$  admits a *crepant* resolution  $V$ , the stringy Euler characteristic of  $X$  may be defined to be the ordinary Euler characteristic  $\chi(V)$  of  $V$ : remarkably, this turns out to be independent of the chosen crepant resolution.

Celestial integration extends this notion to a whole class in  $A_*X$ . By the normalization property (Theorem 3.1 (1)),

$$\left( \int_{\mathcal{C}_X} \mathbb{1}(0) d\mathbf{c}_X \right)_{\text{id}} = c(TX) \cap [X]$$

if  $X$  is nonsingular; but the expression on the left-hand-side defines an element of  $A_*X$  *even if  $X$  is singular* (in fact, the celestial integral defines this expression together with distinguished lifts to all varieties mapping to  $X$ ). If  $X$  admits a crepant resolution  $\pi : V \rightarrow X$ , it is easy to check that this definition produces the push-forward  $\pi_*c(TV) \cap [V]$ . By the Poincaré-Hopf theorem, therefore, the degree of this class recovers the stringy Euler characteristic of  $X$  in this case.

This in fact holds for any  $X$  for which the stringy Euler characteristic is defined, justifying the following:

**Definition 5.1.** The *stringy Chern class* of  $X$  is the identity manifestation

$$\left( \int_{\mathcal{C}_X} \mathbb{1}(0) d\mathbf{c}_X \right)_{\text{id}} .$$

Coincidentally, a notion of stringy Chern class was produced simultaneously as the one presented above, by Tommaso de Fernex, Ernesto Lupercio, Thomas Nevins, and Bernardo Uribe (in fact, the preprint [dFLNU] appeared on the arXiv during the São Carlos/Luminy conference!). While the approaches to the two notions differ somewhat, the two stringy classes agree.

**5.2.** There is a subtlety here, which I can only touch upon in this note. For singular  $X$ , the notion of celestial integral depends on the choice of a good notion of *relative canonical divisor*. The ‘usual’ notion is constructed starting from the *double dual*  $\omega_X$  of the Kähler differentials  $\Omega_X^{\dim X}$  of  $X$ ; this ‘ $\omega$  flavor’ of the celestial integral is what leads to the stringy Chern class recovering the usual stringy Euler characteristic, as explained above, and agreeing with the class introduced by deFernex et al.

The  $\omega$  flavor leads to a technical difficulty, which may make the celestial integral (and hence stringy Chern classes) *undefined* if the singularities of  $X$  are not mild enough—the technical condition is that they should be *log terminal*. Whether stringy classes (or more generally celestial integrals) may be defined for varieties with more general singularities is an open question, see §7.

One way out of this bind is to choose a different notion of relative canonical divisor in the main set-up. For example, one can *avoid* taking the double-dual, leading to the  $\omega$  flavor as mentioned above; this leads to a different notion (which I call the  $\Omega$  flavor of the integral), which is defined for arbitrarily singular varieties. While this yields a stringy Chern class for arbitrary varieties, the meaning of this class (for example vis-a-vis the stringy Euler characteristic) has not been explored.

## 6. CHERN-SCHWARTZ-MACPHERSON CLASSES FROM CELESTIAL INTEGRALS

**6.1.** The stringy notion presented in §5.1 amounts to taking the identity manifestation of the integral of 0 over the whole modification system  $\mathcal{C}_X$  of the variety  $X$ . By the normalization property ((1) in Theorem 3.1), this yields the usual Chern class of the tangent bundle of  $X$  when  $X$  is nonsingular.

There is a different natural way to use the same tool and define a class generalizing  $c(TX) \cap [X]$ : embed  $X$  into an ambient nonsingular variety  $M$ , then compute the identity manifestation of the celestial integral of 0 over the constructible subset  $\mathcal{X}$  determined by  $X$ :

$$\left( \int_{\mathcal{X}} \mathbb{1}(0) d\mathbf{c}_M \right)_{\text{id}} .$$

With due care, this class can be defined in  $A_*X$  (our definition of the celestial integral would only place it in  $A_*M$ ); remarkably, as such it does *not* depend on the ambient variety  $M$ . In fact:

$$\left( \int_{\mathcal{X}} \mathbb{1}(0) d\mathbf{c}_M \right)_{\text{id}} \stackrel{!}{=} c_{\text{SM}}(X) ,$$

the *Chern-Schwartz-MacPherson* class of  $X$ .

This is a famous notion, going back to Marie-Hélène Schwartz ([Sch65]) and Robert MacPherson ([Mac74]). In MacPherson’s construction (as recalled, for example, in [Ful84], §19.1.7), one obtains in fact a natural transformation  $c_*$  from the functor of constructible functions (with proper push-forward defined by Euler characteristic of fibers) to the Chow group functor; applying  $c_*$  to the constant function  $\mathbb{1}_X$  defines the class  $c_{\text{SM}}(X)$ .

**6.2.** The connection between celestial integrals and Chern-Schwartz-MacPherson classes mentioned above goes in fact much deeper. Given any divisor  $\mathcal{D}$  and any constructible subset  $\mathcal{S}$  of a modification system  $\mathcal{C}_X$ , one may define a constructible function by

$$p \mapsto I_X(\mathcal{D}, \mathcal{S}) := \text{deg} \left( \int_{\mathcal{S} \cap p} \mathbb{1}(\mathcal{D}) d\mathbf{c}_X \right) ;$$

here,  $\mathcal{S} \cap p$  is the constructible subset of  $\mathcal{C}_X$  obtained by intersecting  $\mathcal{S}$  with inverse images of  $p$  through the system.

**Theorem 6.1.**

$$\left( \int_{\mathcal{S}} \mathbb{1}(\mathcal{D}) d\mathbf{c}_X \right)_{\text{id}} = c_*(I_X(\mathcal{D}, \mathcal{S})) .$$

Thus, celestial integrals and Chern-Schwartz-MacPherson classes are, in a sense, equivalent information: each can be obtained from the other.

Classes such as the stringy Chern class considered in §5.1 correspond, via MacPherson’s natural transformation, to specific constructible functions. These ‘stringy’ characteristic functions deserve much further study.

The apparatus of Chern-Schwartz-MacPherson classes is an important ingredient in the construction in [dFLNU].

**6.3.** It should be noted that the definition of celestial integration (which I will finally summarize in §7) does *not* rely on Chern-Schwartz-MacPherson classes; the latter are an honest subproduct of the former. Thus, one could try to recover the main defining features of Chern-Schwartz-MacPherson classes from celestial properties.

For example, I would like to venture the guess that *the covariance property of Chern-Schwartz-MacPherson classes is a facet of the change-of-variable formula for celestial integrals*. This should mean that the change-of-variable formula is a Riemann-Roch theorem in disguise. As things stand now I don’t even have a precise version of this ‘guess’ to offer, and I will have to leave it at the stage of half-baked speculations.

## 7. THE DEFINITION

A summary of celestial integration *without* a definition of this notion would be incomplete, even though I have tried to defend the idea that the definition itself is less important than the fact alone that such a notion exists—in practice, the normalization and change-of-variable properties suffice for interesting applications and do not require the actual definition of the integral to be appreciated.

In any case, here is the definition. Given a divisor  $\mathcal{D}$  and a constructible (say closed and proper, for simplicity) subset  $\mathcal{S}$  of the modification system  $\mathcal{C}_X$ , embedded resolution of singularities ensures that there is an object  $\pi : V_\pi \rightarrow X$  in  $\mathcal{C}_X$ , a normal crossing divisors  $E$  with nonsingular components  $E_j$ ,  $j \in J$ , and divisors  $D_\pi$ ,  $S_\pi = \cup_{j \in J_S} E_j$  of  $V_\pi$  such that:

- $\mathcal{D}$  is represented by  $(\pi, D_\pi)$ ;
- $\mathcal{S}$  is represented by  $(\pi, S_\pi)$ ;
- $D_\pi + K_\pi = \sum_{j \in J} m_j E_j$ , with  $m_j \in \mathbb{Q}$ .

This set of data depends on the chosen notion of relative canonical divisor  $K_\pi$ . *Assume that all coefficients  $m_j$  are  $> -1$ .*

**Definition 7.1.**

$$\left( \int_{\mathcal{S}} \mathbb{1}(\mathcal{D}) d\mathcal{C}_X \right)_\pi := c(\Omega_{V_\pi}(\log E)^\vee) \cap \sum_{I \subset J, I \cap J_S \neq \emptyset} \frac{[\cap_{i \in I} E_i]}{\prod_{i \in I} (1 + m_i)}$$

This expression defines the manifestation of the integral on all varieties such as  $V_\pi$ , in which the data  $\mathcal{D}$ ,  $\mathcal{S}$  are ‘resolved’ by a normal-crossing divisor. The manifestation on any other variety is obtained by push-forward, compatibly with the requirement that the celestial integral is an element of the inverse limit  $A_*\mathcal{C}_X$ .

The obvious difficulty with this definition is that it is not at all clear that it should not depend on the chosen  $\pi$  used to resolve the given data. In motivic integration, similar expressions are obtained *a posteriori*, and compute intrinsically defined objects, hence it is clear that they do not depend on the choices. In celestial integration I have to prove the necessary independence explicitly, directly from Definition 7.1.

**Theorem 7.2.** *If all  $m_j$  are  $> -1$ , then the given expression does define an element of the inverse limit  $A_*\mathcal{C}_X$ .*

This is proved by applying the factorization theorem of [AKMW02], which reduces this claim to a computation across blow-ups along nonsingular centers. Manipulating the expressions is a somewhat messy, but manageable, exercise in standard intersection theory.

The independence requires that all  $m_j > -1$  (even though the expression in Definition 7.1 makes sense as soon as no  $m_j$  is  $= -1$ ); this is where the singularities of  $X$  may play a rôle for the particular case  $\mathcal{D} = 0$ , as I discussed in §5.2: the restriction  $m_j > -1$  in this case amounts to the requirement that  $X$  be log terminal.

The difficulty arising if some  $m_j \leq -1$  is that in a chain of varieties connecting two varieties where the data is resolved, one may appear for which the expression in Definition 7.1 does not make sense, for the mundane reason that one of the denominators in the expression may vanish.

This problem arises in many different contexts, of which celestial integration is but one instance (see for example [Vey05], §8, Question I). While there is a feeling that the obstacle is technical rather than conceptual, it has opposed stubborn resistance to the attempts made so far to overcome it, and examples such as the one presented in §3.4 in [Vey] suggest that the issue may be more fundamental than initially expected.

The question of exactly which celestial integrals are well-defined outside the range specified in Theorem 7.2 is subtle and difficult. Answering this question is a worthwhile challenge: the present state of affairs limits the scope of the definition of certain key celestial integrals and hence, as pointed out in §4 and §5, of some potentially interesting applications.

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