PACKING SPHERES AND FRACTAL STRICHARTZ ESTIMATES IN $\mathbb{R}^d$ FOR $d \geq 3$

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January 2005

Fix a dimension $d \geq 2$ and for $x \in \mathbb{R}^d$ and $r > 0$, let $S(x, r)$ stand for the sphere in $\mathbb{R}^d$ with center x and radius r. Identifying the collection of all such spheres with $\mathcal{S} \doteq \mathbb{R}^d \times (0, \infty) \subseteq \mathbb{R}^{d+1}$, it makes sense to talk about the dimension of a (Borel) set of spheres. Since the dimension of any $S(x, r)$ is $d - 1$, it is natural to conjecture that if E is the union of the spheres in a collection whose dimension exceeds one, then $|E|$, the d-dimensional Lebesgue measure of E, is positive. When $d = 2$ this is a difficult question, answered in the affirmative in Wolff's paper [7]. When $d > 2$ we will establish the same result much more easily by giving an elementary proof of the following estimate for the spherical average operator $Tf(x, r) = \int_{\Sigma^{(d-1)}} f(x - r\sigma) d\sigma$.

Theorem 1. *Suppose $d > 2$. Fix $\alpha \in (1, d+1)$ and suppose that μ is a nonnegative Borel measure on a compact subset $\mathcal{K}$ of $\mathcal{S}$. Let*

$$E_\alpha(\mu) = \int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(S_1) d\mu(S_2)}{|S_1 - S_2|^\alpha}$$

denote the α-dimensional energy of μ, where $|S_1 - S_2| = |y_1 - y_2| + |r_1 - r_2|$ when $S_j = S(y_j, r_j)$. Then, for Borel sets $E \subseteq \mathbb{R}^d$, we have the restricted weak type estimate

$$\|T\chi_E\|_{L^{\alpha,\infty}(\mu)} \leq C \; E_\alpha(\mu)^{\frac{1}{2\alpha}} |E|^{\frac{1}{2}}, \tag{1}$$

where the constant depends only on d, α, and the inf *and* sup *of r on $\mathcal{K}$.*

Since any set whose Hausdorff dimension exceeds one carries a measure μ as in Theorem 1 for some $\alpha > 1$, the following corollary is immediate.

Corollary 1. *Suppose $E \subseteq \mathbb{R}^d$ and $\mathcal{T} \subseteq \mathcal{S}$ are Borel sets. Suppose that $\mathcal{T}$ has dimension exceeding one and that for every $(x, r) \in \mathcal{T}$, $S(x, r) \cap E$ has positive $(d-1)$-dimensional measure. Then $|E| > 0$.*

We mention that the paper [2] contains an analogue of Corollary 1 in the case where the projection of $\mathcal{T}$ onto $\mathbb{R}^d$ has dimension greater than one.

It should also be true that for $0 < \alpha < 1$ the union of an α-dimensional set of spheres has dimension at least $d - 1 + \alpha$. We will prove this if the points (x, r) corresponding to the spheres either lie on a curve or comprise an appropriate Cantor

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set. It seems likely that these extra hypotheses are redundant, but we have not yet been able to eliminate them.

Estimates like (1) are closely connected with the wave equation. Thus it is not surprising that (1) implies the following estimate of Strichartz type, again a higher-dimensional analogue of certain results in [7].

Theorem 2. *Suppose α and μ are as in Theorem 1. Suppose $u(x,t)$ is a solution of the wave equation $u_{tt} - \Delta u = 0$ on $\mathbb{R}^d \times [0,\infty)$ with $u(x,0) = g(x)$, $u_t(x,0) = h(x)$. For $q < \alpha$ and $\epsilon > 0$ there is the estimate*

$$\|u\|_{L^q(\mu)} \lesssim \big(\|g\|_{W^{2,(d-1)/2+\epsilon}} + \|h\|_{W^{2,(d-3)/2+\epsilon}}\big).$$

This note is organized as follows: we begin with the proof of Theorem 1. The argument here was first used in [3] and [4], and then, in a form essentially identical to that employed here, in [5]. After the proof of Theorem 1 we sketch proofs for the statements following Corollary 1. We conclude with the proof of Theorem 2.

Proof of Theorem 1:

For small $0 < \delta < r$, let T_δ be the operator which maps f to its average $T_\delta f(x,r)$ over the annulus $A(x,r,\delta)$ of radii $r-\delta$ and $r+\delta$ centered at x. The estimate (1) is obviously a consequence of similar estimates, uniform in δ, for the operators T_δ. A simple-minded strategy, introduced in [3], for obtaining such estimates for operators like T_δ starts from the inequality

$$|E| \geq \sum_{n=1}^{N} |E_n| - \sum\nolimits_{1\leq m<n\leq N} |E_m \cap E_n|, \tag{2}$$

where the E_n's are subsets of E. In the present case, the E_n's will be intersections of E with annuli $A(x_n, r_n, \delta)$. The measures of the $E_m \cap E_n$ will be controlled by the integral $E_\alpha(\mu)$ and the following crude observation (whose proof we include for the sake of completeness).

Lemma 1. *Suppose $0 < r_0 < R_0$. There is a constant $C = C(R_0)$ such that if $0 < \delta < r_0 < r_1, r_2 < R_0$ and $y_1, y_2 \in \mathbb{R}^d$, then*

$$|A(y_1,r_1,\delta) \cap A(y_2,r_2,\delta)| \leq \frac{C\delta^2}{\delta + |y_1 - y_2| + |r_1 - r_2|}. \tag{3}$$

Proof of Lemma 1: Fix y_1 and y_2 and, with no loss of generality, suppose that

$$y_1 = (0,0,\ldots,0) \text{ and } y_2 = (t,0,0,\ldots,0).$$

For $s_1, s_2 > 0$, the spheres

$$S(y_1,s_1) = \{(x_1,\ldots,x_d) : x_1^2 + \cdots + x_d^2 = s_1^2\}$$

and

$$S(y_2,s_2) = \{(x_1,\ldots,x_d) : (x_1-t)^2 + x_2^2 + \cdots + x_d^2 = s_2^2\}$$

intersect in $\emptyset$ or in a $(d-2)$-sphere $\{(x_1^0, x_2, \dots, x_d)\}$ where

$$s_1^2 - s_2^2 = 2tx_1^0 - t^2, x_2^2 + \cdots + x_d^2 = \rho^2, \text{ and } \rho^2 = s_1^2 - (x_1^0)^2.$$

Then

$$x_1^0 = x_1^0(s_1, s_2) = \frac{t^2 + s_1^2 - s_2^2}{2t}$$

and

$$\rho^2 = \rho(s_1, s_2)^2 = \frac{4s_1^2 t^2 - (t^2 + s_1^2 - s_2^2)^2}{4t^2}.$$

Consider the map $(s_1, s_2, \sigma) \mapsto (x_1^0, \rho\sigma)$ of $(0, \infty)^2 \times \Sigma^{(d-2)}$ into $\mathbb{R}^d$ and the corresponding formula

$$\int_0^\infty \int_0^\infty \int_{\Sigma^{(d-2)}} f(x_1^0, \rho\sigma) J(s_1, s_2)\, ds_1\, ds_2\, d\sigma = \int_{\mathbb{R}^d} f. \tag{4}$$

Suppose $S(y_1, s_1) \cap S(y_2, s_2) \neq \emptyset$. If f approximates the function $\chi_{\{s_1 = s_1^0, s_2 = s_2^0\}}$ and c_{d-2} is the volume of the unit $(d-2)$-sphere, it is easy to see that $c_{d-2} J(s_1^0, s_2^0) = c_{d-2} \rho(s_1^0, s_2^0)^{d-2}$. Some algebra then shows that

$$J(s_1, s_2) = \Big(\frac{\sqrt{|s_1 + s_2 + t| \cdot |s_1 + s_2 - t| \cdot |s_1 - s_2 + t| \cdot |s_1 - s_2 - t|}}{2|t|}\Big)^{d-2}. \tag{5}$$

We will use (4) to estimate

$$|A(y_1, r_1, \delta) \cap A(y_2, r_2, \delta)| \leq c_{d-2} \int_{\{|s_1 - r_1| < \delta\}} \int_{\{|s_2 - r_2| < \delta\}} J(s_1, s_2)\, ds_1 ds_2. \tag{6}$$

If $A(y_1, r_1, \delta) \cap A(y_2, r_2, \delta) \neq \emptyset$ then it follows that $|r_1 - r_2| \leq |y_1 - y_2| + 2\delta$. Now recall that $|y_1 - y_2| = |t|$ to observe that if $|t| \leq \delta$, then (3) follows from $r_j < R_0$. On the other hand, if $|t| > \delta$ then $|s_1 - r_1|, |s_2 - r_2| < \delta$ and $|r_1 - r_2| \leq |t| + 2\delta$ give

$$|s_1 - s_2 \pm t| \leq 2\delta + |r_1 - r_2| + |t| \leq 2|t| + 4\delta \leq 6|t|.$$

With (5) this shows that $J(s_1, s_2) \leq C(R_0)$ and so (6) gives (3).

Let r_0 be the inf of r on the (compact) support of μ. The analogue of (1) for T_δ is the inequality

$$\lambda^2 \mu\{T_\delta \chi_E(S) > \lambda\}^{\frac{2}{\alpha}} \leq C E_\alpha(\mu)^{\frac{1}{\alpha}} |E| \tag{7}$$

for $\lambda > 0$ and $0 < \delta < r_0$. Here $T_\delta f(S)$ is the average of f over the annulus $A(y, r, \delta)$ if $S = S(y, r)$. If $T_\delta \chi_E(S) > \lambda$ for $S = S(y, r)$, then $|E \cap A(y, r, \delta)| \geq c\lambda\delta$ where c depends only on r_0. Thus if

$$S_1, \dots, S_N \in \{T_\delta \chi_E(S) > \lambda\} \doteq \mathcal{T},$$

then (2) and Lemma 1 give

$$|E| \geq cN\lambda\delta - \sum_{1 \leq n < m \leq N} \frac{C\delta^2}{|S_n - S_m|}. \tag{8}$$

Control of the sum in (8) is provided by

Lemma 2. *Let μ be as in Theorem 1. Then given $n \in \mathbb{N}$ and a Borel $\mathcal{T} \subseteq \mathcal{S}$ with $\mu(\mathcal{T}) > 0$, one can choose $S_n \in \mathcal{T}$, $1 \leq n \leq N$, such that*

$$\sum_{1 \leq m < n \leq N} \frac{1}{|S_m - S_n|} \leq \frac{\big(E_\alpha(\mu)\big)^{1/\alpha} N^2}{\mu(\mathcal{T})^{2/\alpha}}.$$

Proof of Lemma 2: Suppose $S_1, \dots, S_N$ are chosen independently and at random from the probability space $(\mathcal{T}, \frac{\mu}{\mu(\mathcal{T})})$. Then, for $1 \leq m < n \leq N$,

$$\mathbb{E}\Big(\frac{1}{|S_m - S_n|}\Big) = \frac{1}{\mu(\mathcal{T})^2} \int_{\mathcal{T}} \int_{\mathcal{T}} \frac{d\mu(S_m) d\mu(S_n)}{|S_m - S_n|} \leq$$

$$\frac{1}{\mu(\mathcal{T})^2} \Big(\int_{\mathcal{T}} \int_{\mathcal{T}} d\mu(S_m) d\mu(S_n)\Big)^{1-1/\alpha} \Big(\int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(S_m)\ d\mu(S_n)}{|S_m - S_n|^\alpha}\Big)^{1/\alpha} \leq \frac{\big(E_\alpha(\mu)\big)^{1/\alpha}}{\mu(\mathcal{T})^{2/\alpha}},$$

by the hypothesis on μ. Thus

$$\mathbb{E}\Big(\sum_{1 \leq m < n \leq N} \frac{1}{|S_m - S_n|}\Big) \leq \frac{\big(E_\alpha(\mu)\big)^{1/\alpha} N^2}{\mu(\mathcal{T})^{2/\alpha}}$$

and the lemma follows.

With Lemma 2, (8) becomes

$$|E| \geq cN\lambda\delta - C\delta^2 \big(E_\alpha(\mu)\big)^{1/\alpha} N^2 / \mu(\mathcal{T})^{2/\alpha}. \tag{9}$$

Let $N_0 = c\lambda\mu(\mathcal{T})^{2/\alpha}/(CE_\alpha(\mu)^{1/\alpha}\delta)$. Noting that $N = N_0$ makes the RHS of (9) equal to 0, we consider two cases:

Case I: Assume $N_0 > 10$.
In this case choose $N \in \mathbb{N}$ such that

$$\frac{\lambda c\mu(\mathcal{T})^{2/\alpha}}{2C\big(E_\alpha(\mu)\big)^{1/\alpha}\delta} \geq N \geq \frac{\lambda c\mu(\mathcal{T})^{2/\alpha}}{3C\big(E_\alpha(\mu)\big)^{1/\alpha}\delta}.$$

Then it follows from (9) that

$$|E| \geq c\frac{\lambda c\mu(\mathcal{T})^{2/\alpha}}{3C\big(E_\alpha(\mu)\big)^{1/\alpha}\delta}\lambda\delta - C\delta^2\big(E_\alpha(\mu)\big)^{1/\alpha}\frac{\lambda^2 c^2 \mu(\mathcal{T})^{4/\alpha}}{4(CE_\alpha(\mu)^{1/\alpha})^2\delta^2}\mu(\mathcal{T})^{-2/\alpha} = \kappa\lambda^2\mu(\mathcal{T})^{2/\alpha}/\big(E_\alpha(\mu)\big)^{1/\alpha}$$

for $\kappa = c^2/(12C)$. This gives (7).

Case II: Assume $N_0 \leq 10$.
In this case (unless $\mathcal{T}$ is empty) we estimate

$$|E| \geq c\lambda\delta \geq \frac{\lambda^2 c^2 \mu(\mathcal{T})^{2/\alpha}}{10CE_\alpha(\mu)^{1/\alpha}}$$

which again yields (7), concluding the proof of Theorem 1.

The following result will allow us to deal with the case $0 < \alpha < 1$.

Proposition 1. *Suppose $0 < \gamma \leq \beta \leq \alpha < 1$ and suppose that μ is a finite nonnegative Borel measure on a compact subset $\mathcal{K}$ of $\mathcal{S}$ satisfying the following condition: given $N \in \mathbb{N}$ and Borel $\mathcal{T} \subseteq \mathcal{S}$, one can choose $S_1, \ldots, S_N \in \mathcal{T}$ such that*

$$\sum_{1 \leq m < n \leq N} \frac{1}{|S_m - S_n|} \leq \frac{CN^{(1+2\beta-\gamma)/\beta}}{\mu(\mathcal{T})^{(1+\gamma)/\beta}}. \tag{10}$$

Suppose

$$\frac{1}{p} = \frac{1+\beta-\gamma}{1+2\beta-\gamma}, \quad \frac{1}{q} = \frac{1+\gamma}{1+2\beta-\gamma}, \quad \eta = \frac{1-\gamma}{1+2\beta-\gamma}.$$

Then there is the estimate

$$\|T_\delta \chi_E\|_{L^{q,\infty}(\mu)} \lesssim |E|^{1/p} \delta^{-\eta}$$

for Borel $E \subset \mathbb{R}^d$ and $\delta \in (0,1)$.

It follows from the proof of Lemma 2.15 in [1] that the estimate

$$\|T_\delta \chi_E\|_{L^{q,\infty}_\mu} \lesssim |E|^{1/p} \delta^{-\eta}$$

implies a lower bound of $n - p\eta$ for the Hausdorff dimension of a Borel set containing positive-measure sections of each sphere S in the support of μ. Plugging in the values for p and η which are given in Proposition 1 yields the lower bound

$$n - (1-\gamma)/(1+\beta-\gamma).$$

We will see below that if $\mathcal{T} \subseteq \mathcal{S}$ has dimension $\alpha \in (0,1)$ and either lies on a curve or is a Cantor set, then $\mathcal{T}$ supports measures μ allowing choices of β and γ arbitrarily close to α and so leading to the desired lower bound of $n - 1 + \alpha$ for the dimension of $\cup_{S \in \mathcal{T}} S$. First, though, we indicate the proof of the proposition.

Proof of Proposition 1: The proof follows the proof of Theorem 2 in [5] and is only a slight modification of the proof of Theorem 1. Using (10) instead of Lemma 1, the analogue of (9) is

$$|E| \geq cN\lambda\delta - C\delta^2 N^{(1+2\beta-\gamma)/\beta} \mu(\mathcal{T})^{-(1+\gamma)/\beta}. \tag{11}$$

The two cases are now defined by comparing

$$N_0 \doteq \left(\frac{c\lambda}{C\delta}\right)^{\beta/(1+\beta-\gamma)} \mu(\mathcal{T})^{\frac{1+\gamma}{1+\beta-\gamma}}$$

and 10. In case $N_0 > 10$, choosing N in (11) such that $N_0/2 \geq N \geq N_0/3$ gives

$$|E| \geq \lambda^{\frac{1+2\beta-\gamma}{1+\beta-\gamma}} \delta^{\frac{1-\gamma}{1+\beta-\gamma}} \mu(\mathcal{T})^{\frac{1+\gamma}{1+\beta-\gamma}} \kappa$$

where

$$\kappa = c^{\frac{1+2\beta-\gamma}{1+\beta-\gamma}} C^{\frac{-\beta}{1+\beta-\gamma}} \Big(\frac{1}{3} - \frac{1}{2^{(1+2\beta-\gamma)/\beta}}\Big) > 0.$$

This gives the desired estimate $\lambda\mu(\mathcal{T})^{1/q} \lesssim |E|^{1/p}\delta^{-\eta}$ if $N_0 > 10$. On the other hand, the inequality $N_0 \leq 10$ gives $\lambda\mu(\mathcal{T})^{(1+\gamma)/\beta} \lesssim \delta$ and so

$$\lambda^A \mu(\mathcal{T})^{A(1+\gamma)/\beta} \lesssim \delta^A \tag{12}$$

if $A > 0$. Since $|E| \geq c\lambda\delta$ (unless $\mathcal{T}$ is empty), there is also the inequality

$$\lambda^{1-A} \lesssim |E|^{1-A}\delta^{A-1} \tag{13}$$

as long as $0 < A < 1$. Multiplying (12) and (13) gives $\lambda\mu(\mathcal{T})^{A(1+\gamma)/\beta} \lesssim |E|^{1-A}\delta^{2A-1}$. Then the choice $A = \beta/(1+2\beta-\gamma)$ yields $\lambda\mu(\mathcal{T})^{1/q} \lesssim |E|^{1/p}\delta^{-\eta}$ again, completing the proof of Proposition 1.

When μ is supported on a curve, the following lemma verifies the hypotheses of Proposition 1. Thus it follows from standard facts about Hausdorff dimension that if the Borel set $\mathcal{K} \subseteq \mathcal{S}$ lies on a curve as in Lemma 3, if $E \subseteq \mathbb{R}^d$ is Borel, and if $\cup_{S\in\mathcal{K}} S \subseteq E$, then

$$\alpha \doteq \dim(\mathcal{K}) \in (0,1) \text{ implies } \dim(E) \geq n-1+\alpha. \tag{14}$$

Lemma 3. *Suppose $\alpha \in (0,1)$. Suppose $\widetilde{\mu}$ is a nonnegative measure on a compact interval $J \subseteq \mathbb{R}$ which satisfies the condition*

$$\widetilde{\mu}(I) \lesssim |I|^\alpha$$

for subintervals $I \subseteq J$. Let μ be the image of $\widetilde{\mu}$ under a one-to-one and bi-Lipschitz mapping of J into $\mathcal{S}$. Suppose $0 < \gamma < \beta < \alpha < 1$. Then given $N \in \mathbb{N}$ and Borel $\mathcal{T} \subseteq \mathcal{S}$, one can choose $S_1, \dots, S_N \in \mathcal{T}$ such that

$$\sum_{1\leq m<n\leq N} \frac{1}{|S_m - S_n|} \leq \frac{C(\alpha,\beta,\gamma)N^{(1+2\beta-\gamma)/\beta}}{\mu(\mathcal{T})^{(1+\gamma)/\beta}}.$$

We omit the proof of Lemma 3 since it is technical, of little intrinsic interest, and completely parallel to the proof of Lemma 2 in [5]. The next lemma is an analogue of Lemma 3 in case $\mathcal{K}$ is a Cantor set.

Lemma 4. *Suppose $2 \leq k, l$ are positive integers and set $\alpha = \log k/\log l$. Suppose that $\mathcal{K} \subseteq \mathcal{S}$ supports a Borel measure μ such that $E_{\alpha-\epsilon}(\mu) < \infty$ for $0 < \epsilon < \alpha$ and having the property that for each $M \in \mathbb{N}$, $\mathcal{K}$ is the union of k^M l^{-M}-separated compact sets $\mathcal{K}_m^M$ of diameter $\simeq l^{-M}$ such that $\mu(\mathcal{K}_m^M) = k^{-M}$. Then, given $\beta \in (0,\alpha)$, $N \in \mathbb{N}$, and Borel $\mathcal{T} \subseteq \mathcal{S}$, one can choose $S_1, \dots, S_N \in \mathcal{T}$ such that*

$$\sum_{1\leq m<n\leq N} \frac{1}{|S_m - S_n|} \leq \frac{C(\beta)N^{(1+\beta)/\beta}}{\mu(\mathcal{T})^{(1+\beta)/\beta}}.$$

Again, it follows from Proposition 1 that (14) holds for such Cantor sets $\mathcal{K}$.

Proof of Lemma 4: Fix $\beta, \epsilon \in (0, \alpha)$ such that

$$\frac{1+\alpha+\epsilon}{\alpha} = \frac{1+\beta}{\beta}. \tag{15}$$

For a given $N \in \mathbb{N}$ and Borel $\mathcal{T} \subseteq \mathcal{S}$, choose $M \in \mathbb{N}$ such that

$$N \simeq k^M \mu(\mathcal{T}). \tag{16}$$

Since $\mu(\mathcal{K}_m^M) = k^{-m}$, $\mathcal{T}$ must intersect $\gtrsim N$ of the $\mathcal{K}_m^M$'s. Therefore we can choose $\simeq N$ points $S_m \in \mathcal{T}$ such that distinct S_m's lie in distinct $\mathcal{K}_m^M$'s. Then, using the hypotheses on the $\mathcal{K}_m^M$'s,

$$k^{-2M} \sum_{1 \leq m < n \leq N} \frac{1}{|S_m - S_n|} \lesssim \sum_{1 \leq m < n \leq N} \int_{\mathcal{K}_m^M} \int_{\mathcal{K}_n^M} \frac{d\mu(T_1) d\mu(T_2)}{|T_1 - T_2|} \lesssim$$

$$\sum_{1 \leq m < n \leq N} \int_{\mathcal{K}_m^M} \int_{\mathcal{K}_n^M} \frac{d\mu(T_1) d\mu(T_2)}{|T_1 - T_2|^{\alpha-\epsilon} (l^{-M})^{1-\alpha+\epsilon}} \leq E_{\alpha-\epsilon}(\mu)(l^M)^{1-\alpha+\epsilon}.$$

Since $l = k^{1/\alpha}$, it follows from (16) and (15) that

$$\sum_{1 \leq m < n \leq N} \frac{1}{|S_m - S_n|} \lesssim (k^M)^{(1+\alpha+\epsilon)/\alpha} \simeq \Big(\frac{N}{\mu(\mathcal{T})}\Big)^{(1+\alpha+\epsilon)/\alpha} = \Big(\frac{N}{\mu(\mathcal{T})}\Big)^{(1+\beta)/\beta}$$

as desired.

We will give the proof of Theorem 2 in case d is even. The proof for odd d is similar but slightly less complicated. Before beginning, we recall the well-known formula for the solution of the wave equation for even d:

$$u(x,t) = \frac{1}{\gamma_d} \Bigg[\Big(\frac{\partial}{\partial t}\Big) \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(d-2)/2} \Big(t^{d-1} \int_{B(0,1)} \frac{g(x+ty)}{\sqrt{1-|y|^2}} dy \Big) +$$

$$\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(d-2)/2} \Big(t^{d-1} \int_{B(0,1)} \frac{h(x+ty)}{\sqrt{1-|y|^2}} dy \Big) \Bigg]$$

for some constant γ_d.

Proof of Theorem 4: Recalling that $Tf(x,t) = \int_{\Sigma^{(d-1)}} f(x+t\sigma) d\sigma$, we define

$$Sf(x,t) = \int_{B(0,1)} f(x+ty) \frac{dy}{\sqrt{1-|y|^2}}.$$

Let t_0 and T_0 be the inf and sup of t on the support of μ. For $\beta > 0$, let I_β be a potential operator on $\mathbb{R}^d$ with smooth and nonzero multiplier $p_\beta(\xi)$ equal to $|\xi|^{-\beta}$ for $|\xi| \geq 1/(2T_0)$. The next lemma is essentially the difference between the proofs of Theorem 2 for even and odd d.

Lemma 5. *If μ is as in Theorem 1 then for $q<\alpha$ and $0<\beta<1/2$ we have*

$$\|Sf\|_{L^q(\mu)} \lesssim \|I_\beta f\|_{L^{2,1}(\mathbb{R}^d)}. \tag{17}$$

Proof of Lemma 5: Suppose $\gamma(\xi)$ is a smooth function on $\mathbb{R}^d$ which vanishes on $B(0,1)$ and which equals $|\xi|^{-d/2}$ for large ξ. Let $m(\xi)$ be the multiplier $e^{2\pi i|\xi|}\gamma(\xi)$ and suppose that K is the kernel on $\mathbb{R}^d$ whose Fourier transform is m. Define the operator S_0 by

$$S_0 f(x,t) = \int_{\mathbb{R}^d} f(x+ty)\ K(y)\ dy.$$

By the asymptotic expansion of the Fourier transform of $(1-|y|^2)_+^{1/2}$, the operator S is the sum of two operators like S_0 and nicer terms. We will explain why

$$\|S_0 f\|_{L^q(\mu)} \lesssim \|I_\beta f\|_{L^{2,1}(\mathbb{R}^d)}. \tag{18}$$

Define $\widetilde{m}$ by

$$p_\beta(\xi)\widetilde{m}(\xi) = m(\xi)$$

so that $\widetilde{m}(\xi) = e^{2\pi i|\xi|}\widetilde{\gamma}(\xi)$ where $\widetilde{\gamma}(\xi)$ is smooth and equal to $|\xi|^{\beta-d/2}$ for large ξ. Let the kernel $\widetilde{K}$ have Fourier transform $\widetilde{m}$ and put

$$\widetilde{S}f(x,t) = \int_{\mathbb{R}^d} f(x+ty)\ \widetilde{K}(y)\ dy.$$

Since

$$\widehat{S_0 f(\cdot,t)}(\xi) = m(t\xi)\widehat{f}(\xi) = p_\beta(t\xi)\widetilde{m}(t\xi)\widehat{f}(\xi) =$$
$$t^{-\beta}\widetilde{m}(t\xi)p_\beta(\xi)\widehat{f}(\xi) = t^{-\beta}\widehat{\widetilde{S}I_\beta f(\cdot,t)}(\xi),$$

(18) will follow from $t_0 \le t \le T_0$ on the support of μ and

$$\|\widetilde{S}f\|_{L^q(\mu)} \lesssim \|f\|_{L^{2,1}(\mathbb{R}^d)}. \tag{19}$$

Let $\epsilon = 1/2-\beta$. According to the considerations on p. 426 of [6], the kernel $\widetilde{K}$ satisfies $|\widetilde{K}(y)| \simeq |1-|y|\ |^{-1+\epsilon}$ for $|y|\le 2$ and $|\widetilde{K}(y)| \lesssim |y|^{-(d+1)/2}$ for large y. Thus

$$\widetilde{K} = \sum_{j=0}^{\infty} \widetilde{K}_j$$

where $\widetilde{K}_0 \in L^2(\mathbb{R}^d)$ and for $j=1,2,\ldots$

$$|\widetilde{K}_j| \lesssim 2^{-j\epsilon}2^j\chi_{A(0,1,2^{-j})}.$$

(We recall that $A(0,1,2^{-j})$ is the annulus centered at 0 with radii $1-2^{-j}$ and $1+2^{-j}$.) If the operators $\widetilde{S}_j$ correspond to the kernels $\widetilde{K}_j$, then $\widetilde{S}_0$ maps $L^2(\mathbb{R}^d)$ into $L^\infty(\mu)$ (since t is bounded away from 0 on the support of μ). Also, for $j=1,2,\ldots$ we have

$$\|\widetilde{S}_j f\|_{L^q(\mu)} \lesssim 2^{-j\epsilon}\|T_{2^{-j}}f\|_{L^q(\mu)} \lesssim 2^{-j\epsilon}\|f\|_{L^{2,1}(\mathbb{R}^d)}$$

by the analogue of Theorem 1 for the operators T_δ. Since $\widetilde{S} = \sum \widetilde{S}_j$, (19) follows.

Returning to the proof of Theorem 4, we can write

$$u(x,t) = \sum_{|\alpha|=d/2} c_\alpha(t) S(D_\alpha g)(x,t) + \sum_{|\alpha|=d/2-1} d_\alpha(t) S(D_\alpha h)(x,t)$$

where the coefficients c_α and d_α are bounded on the support of μ. Fix $\epsilon > 0$ and let $\beta = 1/2 - \epsilon/2$. If $|\alpha| = d/2$ then Lemma 5 gives

$$\|S(D_\alpha g)\|_{L^q(\mu)} \lesssim \|I_\beta D_\alpha g\|_{L^{2,1}(\mathbb{R}^d)} \lesssim \|I_\beta D_\alpha g\|_{W^{2,\epsilon/2}} \lesssim \|g\|_{W^{2,d/2+\epsilon/2-\beta}} = \|g\|_{W^{2,(d-1)/2+\epsilon}}.$$

Analogous considerations for h complete the proof of Theorem 2.

References

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