3-Manifolds that are covered by two open Bundles

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Abstract

We obtain a list of all closed 3-manifolds that are covered by two open submanifolds, each homeomorphic to an open disk bundle over S^1, or an open I-bundle over the 2-sphere, the projective plane, the torus, or the Klein bottle.[1] [2]

0 Introduction

The F-category $F(M)$ of a closed 3-manifold M is the minimum number of critical points of smooth functions $M \longrightarrow R$. A lower bound for $F(M)$ is the Lusternik-Schnirelmann category $cat\,(M)$ of M, which is a homotopy invariant and is defined to be the smallest number of sets, open and contractibe in M, needed to cover M. An invariant that turns out to be equivalent to $F(M)$ is the smallest number $C(M)$ of open balls needed to cover M. Note that $2 \leq C(M), F(M), cat\ (M) \leq 4$ and denote by $\mathcal{B}$ a connected sum of any number of S^2-bundles over S^1. Then the results about these three invariants can be summarized as follows:

$F(M) = 2 \Leftrightarrow M = S^3$, $F(M) \leq 3 \Leftrightarrow M = \mathcal{B}$ (proved in [12]).

$C(M) = 2 \Leftrightarrow M = S^3$, $C(M) \leq 3 \Leftrightarrow M = \mathcal{B}$ (proved in [8]).

$cat\,(M) = 2 \Leftrightarrow M \simeq S^3$, $cat\,(M) \leq 3 \Leftrightarrow M \simeq \mathcal{B}$ (proved in [3]).

(Here $\simeq$ denotes homotopy equivalence).

Generalization of these invariants were introduced by Clapp and Puppe [1] and Khimshiashvili and Siersma [9]: Let A be a closed k-manifold, $0 \leq k \leq 2$. A subset G in the 3-manifold M is A - *categorical* if the inclusion map $i : G \longrightarrow M$ factors homotopically through A. An A-*function* on M is a smooth function $M \longrightarrow R$ whose critical set is a finite disjoint union of components each diffeomorphic to A. The A-*category* $cat_A\,(M)$ of M is the smallest number

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of sets, open and A-categorical needed to cover M. The A*-complexity* $F_A(M)$ of M is the minimum number of components of the critical set over all A-functions on M.

Then $cat_{\text{point}}(M) = cat(M)$, $F_{\text{point}}(M) = F(M)$, $cat_{S^1}(M)$ is the *round category* of M, and $F_{S^1}(M)$ is the *round complexity* of M, studied in [9].

It is now natural to ask about minimal covers of M by open sets, each homotopy equivalent to A. In particular when A is a point, S^1, or a closed 2–manifold, consider covers of M by open disk bundles over A, i.e. open 3-balls, $\overset{\circ}{D}{}^2$-bundles over S^1, and $\overset{\circ}{I}$-bundles over surfaces. For such an open disk bundle $B(A)$ over A let $C_{B(A)}(M)$ denote the minimal number of sets, each homeomorphic to $B(A)$, needed to cover M. In this paper we classify all closed 3-manifolds for which $C_{B(A)}(M) = 2$, where A is S^1, S^2, the projective plane P^2, the torus T, or the Klein bottle K. (Note that $C_{B(point)}(M) = C(M)$). The results are summarized in a table at the end of the paper. Some results are unexpected; for example the manifolds for which $C_{T\times\overset{\circ}{I}}(M) = 2$ include all lens spaces (including S^3), which can be seen as follows. Let $L_1 = l_1 \cup l_2$ be the Hopf link in S^3 and let l_i' be parallel to l_i so that $L_2 = l_1' \cup l_2'$ is a Hopf link disjoint to L_1. Then $S^3 = \left(S^3 - L_1\right) \cup \left(S^3 - L_2\right)$ is a union of two open $T \times \overset{\circ}{I}$'s. A similar construction can be made for any lens space.

1 Preliminaries

Throughout this paper we work in the PL-category. Our goal is to obtain information about closed 3-manifolds that are covered by open sets each of which is homeomorphic to the interior of a compact 3-manifold. Our main lemma shows that we can reduce the problem of a covering by two open sets to a canonical covering by two compact manifolds, each pl embedded.

1.1 (Main Lemma) Suppose M is a closed 3-manifold covered by two open sets H_1, H_2 such that H_i is homeomorphic to the interior of a compact connected 3-manifold V_i $(i = 1, 2)$. Then M admits a covering $M = V_1 \cup V_2$ such that $\partial V_1 \cap \partial V_2 = \emptyset$ and V_1, V_2 are pl embedded.

Proof. Using collars on ∂V_i $(i = 1, 2)$ we can write $H_i = \bigcup_{k=1}^{\infty} intV_k^{(i)}$, where $V_k^{(i)} \approx V_i$, $V_k^{(i)} \subset intV_{k+1}^{(i)}$, $k = 1, 2, \ldots$ The complement H_1^c of H_1 in M is a compact subspace of H_2 and it follows that $H_1^c \subset intV_n^{(2)}$ for some n. Now $\left(intV_n^{(2)}\right)^c$ is a compact subspace of H_1 and hence $\left(intV_n^{(2)}\right)^c \subset intV_m^{(1)}$ for some m. Note that $\partial V_n^{(2)} \subset \left(intV_n^{(2)}\right)^c \subset V_m^{(1)}$. Hence if we let $V_1 = V_m^{(1)}$ and $V_2 = V_n^{(2)}$ in M we obtain $M = V_1 \cup V_2$ as desired. ∎

By a *knot space* we mean a 3-manifold N homeomorphic to the complement of the interior of a regular neighborhood of a non-trivial knot in S^3. Note that ∂N contains a meridian curve C so that attaching a 2-handle to N with core along C yields B^3. The next lemma is well-known.

1.2 Lemma Suppose M is a compact irreducible 3-manifold.

(i) If M contains a 2-sided compressible torus T then either T bounds a solid torus or a knot space N in M with an essential curve of ∂N bounding a disk in $\overline{M-N}$. If T is a compressible boundary component of M then $M = D^2 \times S^1$.

(ii) If M contains a 2-sided compressible Klein bottle K then either K bounds a solid Klein bottle in M or M contains a 2-sided projective plane P^2. If K is a compressible boundary component then M is a solid Klein bottle.

Proof.
(i) Let $D \times [-1,1]$ be a neighborhood of a compressing disk $D = D \times \{0\}$ with $D \times [-1,1] \cap T = \partial D \times [-1,1] \cap T$. The sphere $S = (T - D \times [-1,1] \cap T) \cup D \times \{-1\} \cup D \times \{1\}$ bounds a ball B in M. If $D \cap B = \emptyset$ then $B \cup D \times [-1,1]$ is a solid torus in M bounded by T. If $D \subset B$ then $T \subset B$ such that $\partial B \cap T$ is an essential annulus of T. Hence $\overline{B - D \times [-1,1]}$ is a knot space (or a solid torus) in M bounded by T.

(ii) If we surger K as above along a compressing disk D we obtain a 2-sphere S if ∂D does not separate K. Then $B \cup D \times [-1,1]$ is a solid Klein bottle bounded by K. (The case $D \subset B$ can not happen since a Klein bottle does not imbed in a ball). If ∂D separates K into two moebius bands then $(K - D \times [-1,1] \cap K) \cup D \times \{-1\} \cup D \times \{1\}$ gives two 2-sided P^2's in M. ■

Notation.
By $B \tilde{\times} F$ we denote a twisted F-bundle over B, not homeomorphic to $B \times F$. In particular, $S^1 \tilde{\times} D^2$ is the solid Klein bottle, $S^1 \tilde{\times} S^2$ is the non-orientable S^2 -bundle over S^1, and $P^2 \tilde{\times} I$ is the once-punctured projective space P^3. The twisted I-bundles over a torus T and a Klein bottle K are described in the next section.

The union of two 3-manifolds N_1, N_2 glued together along boundary components is denoted by $N_1 \cup_\partial N_2$.

L denotes any lens space (including S^3 and $S^1 \times S^2$).

$S\,(2,2,n)$ denotes a Seifert fiber space over the 2-sphere with three exceptional fibers of orders $2,2,n$ ($n \geq 0$).

The symbol $\sim$ means *homologous to.*

The symbol $\approx$ means *homeomorphic.*

2 I-bundles and (semi)-bundles over the torus and Klein bottle

Recall that an I-bundle over a surface F is *twisted* if it is not the product I-bundle $F \times I$. The twisted I-bundle $a^2 \widetilde{\times} I$ over the annulus a^2 is homeomorphic to the product I-bundle $m^2 \times I$ over the moebius band m^2. The twisted I-bundle $m^2 \widetilde{\times} I$ over m^2 is homeomorphic to the solid torus $D^2 \times I$ (with m^2 embedded in $D^2 \times I$ so that ∂m^2 is a $(1,2)$-curve on $\partial D^2 \times S^1$).

(2.1) There is only one twisted I-bundle $T \widetilde{\times} I = m^2 \times S^1$ over the torus $T = S^1 \times S^1$.

To see this, note that in such an I-bundle N there is a simple closed curve c on T such that the restriction of the I-bundle over c is a moebius band. Now c cuts T into an annulus a^2 and the restriction of the I-bundle over a^2 is twisted. Hence N is the quotient $m^2 \times I / (x,0) \sim (\varphi(x), 1)$ for a homeomorphism φ of m^2. If φ is isotopic to the identity then $N = m^2 \times S^1$. The case that φ is not isotopic to the identity can not happen since then φ would reverse on orientation of ∂m^2 which would cause ∂N to be a Klein bottle; but ∂N is a torus since it is 2-sheeted cover of T.

(2.2) There are exactly two twisted I-bundles over the Klein bottle $K = S^1 \widetilde{\times} S^1$.

These can be described as follows. The restriction of such an I-bundle N over a separating simple closed curve on K splits N into two I-bundles over moebius bands m_1^2, m_2^2, at least one of which is twisted. There are two possibilities.

(i) $N = m_1^2 \widetilde{\times} I \cup m_2^2 \widetilde{\times} I$ is a union of two solid tori along an annulus in their boundary and N can be described as a Seifert fiber space with orbit a disk and two exceptional fibers of order 2. In this case N is orientable and is denoted by $\left(K \widetilde{\times} I\right)_0$.

(ii) $N = m_1^2 \times I \cup m_2^2 \widetilde{\times} I$, where $\partial m_1^2 \times I$ is identified with an annular neighborhood of ∂m_2^2 in $\partial D^2 \times S^1 = \partial\left(m_2^2 \widetilde{\times} I\right)$. In this case ∂N is a Klein bottle and we denote this I-bundle over K by $\left(K \widetilde{\times} I\right)_{N_0}$.

Another description of $\left(K \widetilde{\times} I\right)_{N_0}$ is obtained by cutting K along a 2-sided non-separating curve into an annulus. As for $T \widetilde{\times} I$ we obtain $\left(K \widetilde{\times} I\right)_{N_0}$ as the quotient $m^2 \times I / (x,0) \sim (\psi(x), 1)$, where ψ is not isotopic to the identity. Viewing m^2 as a rectangle with a pair of opposite edges identified, ψ is induced by a reflection about a line mid-way betwen the two edges (cf [10]). Thus $\left(K \widetilde{\times} I\right)_{N_0} \approx S^1 \widetilde{\times} m^2$, the twisted m^2-bundle over S^1.

Following Hatcher [4] we call a union of two twisted I-bundles over a torus T (resp. Klein bottle K) glued together along their boundary component a torus (resp. Klein bottle) *semi-bundle.* These semi-bundles are essentially classified by the isotopy classes of the gluing maps (see e.g. [4, Thm 5.1]).

There are exactly four isotopy classes of homeomorphisms of the Klein bottle ([10]) that lead to exactly four Klein bottle-bundles over S^1, described in [6].

3 Covers by $intM_1$ and $intM_2$

In this and the following sections we consider a closed 3-manifold M that is covered by two open sets $intM_1$, $intM_2$ where M_1, M_2 are compact connected 3-manifolds. By the Main Lemma we assume throughout that

$$M = M_1 \cup M_2,\ M_1 \approx M_2 \text{ compact, } \partial M_1 \cap \partial M_2 = \emptyset. \tag{*}$$

We let $Q = M_1 \cap M_2 \subset M$. Note that the boundary of each component of Q contains a component of both ∂M_1 and ∂M_2. We observe

(i) If M_1, M_2 are irreducible then $\overline{M_i - Q}$ is irreducible ($i = 1, 2$).

For a 2-sphere in $int\left(\overline{M_1 - Q}\right)$ bounds a ball B in $int\,(M_1)$. If B does not lie in $\overline{M_1 - Q}$ then B contains a component of Q, hence a component of ∂M_1, a contradiction.

(ii) If M_1, M_2 are irreducible and $M \neq S^3$ then Q is irreducible.

For a 2-sphere S in Q bounds balls $B_1 \subset M_1$, $B_2 \subset M_2$. Either $B_1 = B_2 \subset Q$ or $B_1 \cap B_2 = S$ and $M = B_1 \cup_\partial B_2 = S^3$.

3.1 Covers by open balls and open disk bundles over S^1

(a) If $M_i \approx B^3$ then $M = S^3$.

Proof. ∂M_2 bounds a ball B in M_1 and $M = M_2 \cup_\partial B = S^3$. ■

(b) If $M_i = S^1 \times D^2$ then $M = L$.

Proof. Since M_1 does not contain a closed incompressible surface there is a compressing disk D for ∂M_2 in M_1. If $D \subset \overline{M_1 - Q}$ then $\overline{M_1 - Q}$ is a solid torus (by Lemma 1.2(i) and 3(i)) and $M = \overline{M_1 - Q} \cup_\partial M_2$ is a lens space.

If $D \subset Q$ then viewing a regular neighborhood of D in Q as a 2-handle $U\,(D)$ we get $\overline{M_1 - Q} \cup U\,(D) \subset M_1$ bounded by a 2-sphere. Hence $M = \left(\overline{M_1 - Q} \cup U\,(D)\right) \cup_\partial \left(\overline{M_2 - U\,(D)}\right)$ is a union of two balls, i.e. $M = S^3$. ■

(c) If $M_i \approx S^1\tilde{\times}D^2$ then $M = S^1\tilde{\times}S^2$.

Proof. ∂M_2 is compressible in M_1 and M_1 does not contain a projective plane. By Lemma 1.2(ii), ∂M_2 bounds a solid Klein bottle $M_1' \subset M_1$ and $M = M_1' \cup_\partial M_2 = S^2\tilde{\times}S^1$ (see e.g. [7, 2.14]). ∎

3.2 Covers by open I-bundles over S^2 or P^2

(a) If $M_i \approx S^2 \times I$ then $M = S^3$, $S^1 \times S^2$ or $S^1\tilde{\times}S^2$.

Proof. Let $\partial M_2 = S_0 \cup S_1 \subset intM_1$.

If S_0 bounds a ball B_0 in M_1 then $B_0 \subset \overline{M_1 - Q}$ since M is closed. Now $M_2' = M_2 \cup_\partial B_0$ is a ball and $M = M_1 \cup M_2'$. The boundary S_1 of M_2' is not isotopic to a boundary sphere of M_1 (since M is closed) and hence bounds a ball B_1 in M_1, different from M_2' and $M = M_2' \cup_\partial B_1 = S^3$.

If both S_0 and S_1 are parallel to the boundary spheres of M_1 then S_0 and S_1 bound a submanifold $M_2' \approx S^2 \times I$ in M_1 and we obtain $M = \overline{M_1 - M_2'} \cup_\partial M_2'$, hence $M = S^1 \times S^2$ or $S^1\tilde{\times}S^2$. ∎

(b) If $M_i = P^2 \times I$ then $M = P^2 \times S^1$.

Proof. This follows from the fact that any projective plane in M_1 is isotopic to a boundary component, hence $M \approx M_1 \cup_\partial M_2$. (Note that there is no twisted P^2-bundle over S^1). ∎

(c) If $M_i = P^2\tilde{\times}I$ then $M = P^3$ or $P^3\#P^3$.

Proof. If ∂M_2 bounds a ball B in M_1 then $M = M_2 \cup_\partial B = P^3$. Otherwise ∂M_2 is parallel in M_1 to ∂M_1 and $M \approx M_1 \cup_\partial M_2 = P^3\#P^3$. ∎

3.3 Covers by open I-bundles over $S^1 \times S^1$ and $S^1\tilde{\times}S^1$.

Let $T = S^1 \times S^1$ and $K = S^1\tilde{\times}S^1$.

(a) If $M_i = T \times I$ then $M = L$ or a T-bundle over S^1.

Proof. Let $\partial M_2 = T_0 \cup T_1$, $\partial M_1 = T_0' \cup T_1'$.

If T_0 is incompressible in M_1 then it is isotopic to a component of ∂M_1 and splits M_1 into two copies M_1', M_1'' . Assume $T_0' \subset M_1''$, $T_1' \subset M_1'$. Then $T_1 \subset M_1'$, say. Then (since $T_0', T_0 \subset \partial Q$ and $T_0' \subset intM_2$) it follows that M_1'' is a component of $Q \subset M_1$. The other component(s) of Q are in M_1' and are bounded by T_1' and T_1. Since $T_1' \not\sim 0$ in M_1' there is exactly one component P of Q in M_1' bounded by T_1' and T_1. Hence $T_1 \not\sim 0$ in M_1' and Lemma 1.2(i) implies that T_1 is incompressible in M_1'. Hence T_0, T_1 are isotopic in M_1 to T_0', T_1' and it follows that $M \approx M_1 \cup_\partial M_2$ is a T-bundle over S^1.

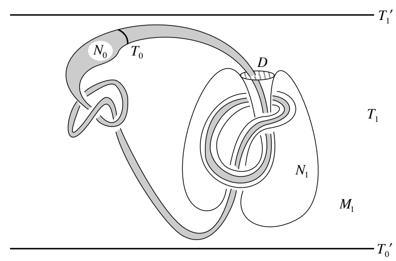

Figure 1:

Now suppose that T_0, T_1 are both compressible in M_1, hence, by Lemma 1.2(i), T_i bounds a solid torus or knot space N_i in M_1 (i=0,1). Now T_1 is not contained in N_0. Otherwise an arc in M_2 from a point of T_0 to a point of T_1 would be in N_0 (since T_0 separates in M_1), and it would follow that $M_2 \subset N_0 \subset M_1$, a contradiction. Similarly T_0 is not contained in N_1; hence N_0 and N_1 are disjoint. If N_0 is a solid torus then $M_2' = M_2 \cup_\partial N_0$ is a solid torus and $M = M_2' \cup_\partial N_1$. Thus if N_1 is also a solid torus, M is a lens space. If N_1 is a knot space then a meridian curve on ∂N_1 bounds a compressing disk D for M_2' in $\overline{M_1 - N_1}$ (see Figure 1).

For a regular neighborhood $U(D)$ in $\overline{M_1 - N_1}$ we obtain $M = \overline{M_2' - U(D)}$ $\cup_\partial \overline{N_1 \cup U(D)}$ a union of two balls, hence $M = S^3$.

The case that both N_0 and N_1 are knot spaces in M_1 can not happen. For in this case a compressing disk D for T_1 in $\overline{M_1 - N_1}$ must intersect N_0, since otherwise D would be a compressing disk for T_1 in M_2. But then an essential innermost curve of $T_0 \cap D$ bounds a disk D' on D which would be a compressing disk for T_0 in N_0 or in M_2, a contradiction. ■

(b) If $M_i = K \times I$ then $M = S^1 \tilde{\times} S^2$ or a K-bundle over S^1.

Proof. Let $\partial M_2 = K_0 \cup K_1 \subset int M_1$.

If K_0 is compressible in M_1 it bounds a solid Klein bottle V_0 in M_1 (by Lemma 1.2(ii), since M_1 does not contain P^2's). The same argument as in case (a) shows that K_1 is also compressible and bounds a solid Klein bottle V_1 in M_1 such that V_0 and V_1 are disjoint. Then $M = (M_2 \cup_\partial V_0) \cup_\partial V_1$ is a union of two solid Klein bottles, hence $M = S^1 \tilde{\times} S^2$.

If both K_0, K_1 are incompressible in M_1 then they are boundary parallel and $M = M_1 \cup M_2$ is a K–bundle over S^1. ■

We next consider the cases of twisted I-bundles over T and K.

3.3.1 Lemma Let M_i be a twisted I-bundle over T or K ($i = 1, 2$).

(i) If ∂M_1 is incompressible in M_2 then $M \approx M_1 \cup_\partial M_2$ is a semi-bundle.

(ii) If ∂M_1 is compressible in M_2 then $M = M_2 \cup_\partial (S^1 \times D^2)$ (for $M_i = T\tilde{\times}I$ or $(K\tilde{\times}I)_0$), resp. $M = M_2 \cup_\partial (S^1\tilde{\times}D^2)$, (for $M_i = (K\tilde{\times}I)_{N_0}$).

Proof. If ∂M_1 is incompressible M_2 then it is parallel to ∂M_2 in M_2 and $M \approx M_1 \cup_\partial M_2$.

If ∂M_1 compresses in M_2 then it bounds a solid torus, a knot space, or a solid Klein bottle in M_2 (by Lemma 1.2). It can not bound a knot space N since otherwise a meridian of ∂N would bound a compressing disk D in $\overline{M_2 - N} \subset Q$ and hence D would be a compressing disk for ∂M_1 in M_1. It follows that $M = M_2 \cup_\partial (S^1 \times D^2)$ or $M_2 \cup_\partial (S^1\tilde{\times}D^2)$. ■

(c) If $M_i = T\tilde{\times}I$ then M is a torus semi-bundle or $M = P^2 \times S^1$ or $M = S^1\tilde{\times}S^2$.

Proof. By the previous lemma it suffices to consider the case that $M = M_2 \cup_\partial (S^1 \times D^2)$.

In the 2-sheeted orientable cover $\widetilde{M}$ of M, $M_2 = m^2 \times S^1$ lifts to $a^2 \times S^1 = T \times I$ and the attaching solid torus $S^1 \times D^2$ lifts to two attaching solid tori. Hence $\widetilde{M}$ is a lens space; its fundamental group is infinite, since it covers the closed non-orientable manifold M. By the classification of (orientation-reversing) fixed point free involutions on $S^1 \times S^2$ ([13], [14, Corollary]) M is as claimed. ■

(d) If $M_i = (K\tilde{\times}I)_0$ then M is a Klein bottle semi-bundle or $M = P^3\#P^3$ or $M = S(2, 2, n)$ (for any $n \geq 0$).

Proof. Again we need to consider only the case that $M = M_2 \cup_\partial (S^1 \times D^2)$. Writing M_2 as a Seifert fiber space over a disk with two exceptional fibers each of order 2 we obtain $M = S(2, 2, n)$ if the meridian ∂D^2 of the attaching solid torus is not homotopic to a fiber on ∂M_2 and $M = P^3\#P^3$ otherwise (see e.g. [5]). ■

(e) If $M_i = (K\tilde{\times}I)_{N_0}$ then M is a Klein bottle semi-bundle or $M = P^2 \times S^1$.

Proof. Considering only the case that $M = M_2 \cup_\partial (S^1\tilde{\times}D^2)$ we represent $M_2 = S^1\tilde{\times}m^2$ (as in section 2) and note that ∂m^2 cuts $\partial M_2 = S^1\tilde{\times}\partial m^2$ into an annulus. Up to isotopy there is only one simple closed curve on K that cuts K into an annulus ([10]). Thus there is only one way to attach $S^1\tilde{\times}D^2$ to M_2 : the meridian ∂D^2 of $S^1\tilde{\times}D^2$ must be glued to ∂m^2 and it follows that $M = (S^1\tilde{\times}m^2) \cup_\partial (S^1\tilde{\times}D^2) = S^1\tilde{\times}P^2 = S^1 \times P^2$. ■

Figure 2 shows that $P^2 \times S^1$ admits indeed a decomposition of type $(K\tilde{\times}I)_{N_0} \cup_\partial (S^1\tilde{\times}D^2)$.

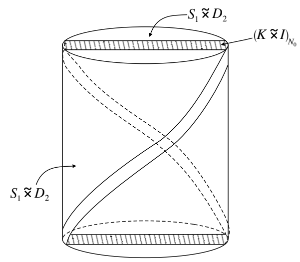

Figure 2:

The following table summarizes the results.

$$M = intM_1 \cup intM_2$$

M_i	B^3	$S^1 \times D^2$	$S^1 \widetilde{\times} D^2$	$S^2 \times I$	$P^2 \times I$	$P^2 \widetilde{\times} I$
M	S^3	L	$S^1 \widetilde{\times} S^2$	S^3 $S^1 \times S^2$ $S^1 \widetilde{\times} S^2$	$P^2 \times S^1$	P^3 $P^3 \# P^3$

M_i	$T \times I$	$T \widetilde{\times} I$	$K \times I$	$\left(K \widetilde{\times} I\right)_0$	$\left(K \widetilde{\times} I\right)_{N_0}$
M	L T-bundles over S^1	$S^1 \widetilde{\times} S^2$ $P^2 \times S^1$ T-semi bundles	$S^1 \widetilde{\times} S^2$ K-bundles over S^1	$P^3 \# P^3$ $S(2, 2, n)$ K-semi bundles (orientable)	$P^2 \times S^1$ K-semi bundles (non orientable)

Conversely it is easy to see that each manifold in the table is a union of two open covers as indicated.

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