Growth of A-Harmonic Functions and Carnot Groups

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November 5, 2005

Abstract

The order of growth of a harmonic function is determined by the growth of its gradient and conversely. We extend these results to solutions of certain subelliptic equations in John domains in Carnot groups. The modulus of the gradient is replaced by a local average of the horizontal gradient. In the harmonic case these quantities are equivalent. The proof uses recent integral inequalities associated with work on potential theory in Carnot groups. We also obtain results on the mutual growth of related A-harmonic functions which generalize corresponding results for conjugate harmonic functions.

Primary 35J60, 30G30, Secondary 35J70 Keywords : Order of growth, A-harmonic funtions, Carnot groups

1 Introduction

The following seminorm measures the rate of growth of a function f in the unit disk $\mathbb{D}$ of the complex plane.

For $\alpha < 0$ and $f : \mathbb{D} \to \mathbb{R}^n$:

$$O_{\alpha,\mathbb{D}}(f) = \inf\{m : |f(z_1) - f(z_2)| \leq m(\min_{j=1,2}(1 - |z_j|))^{\alpha}, z_1, z_2 \in \mathbb{D}\},$$

Theorem 1.1 *Suppose that u is harmonic in the unit disk* $\mathbb{D}$ *of the complex plane. The following are equivalent :*

a) $O_{\alpha,\mathbb{D}}(u) < \infty$.

b) $|\nabla u(z)| \leq C(1 - |z|)^{\alpha-1}$ *for some constant* C *independent of* u.

The implication a) implies b) follows from the Poisson integral formula while b) implies a) can be obtained by integrating over hyperbolic geodesics.

The main results of this paper appear in Section 6. Theorem 6.1 is a local version of Theorem 1.1 for A-harmonic functions in domains in Carnot groups. We use a local average of the horizontal gradient in general. When u is harmonic, this is equivalent to the modulus of the gradient at the same point. We obtain a global result in John domains presented in Theorem 6.2. We obtain results on the mutual growth of related A-harmonic functions on Carnot groups in Theorems 6.3 and 6.4. These results apply also to conjugate harmonic functions in the plane. We give examples in the plane of p,q-harmonic functions which show that, in these cases, our results are in many ways best possible. In Section 2 we describe Carnot groups and introduce notation. Section 3 is devoted to certain domains in general metric spaces. The results here and the proofs are adapted from the Euclidean case with assumptions on the geometry of metric balls. John domains and Ord_α-extension domains are the natural domains for the study of the growth of A-harmonic functions.

We introduce subelliptic equations and corresponding integral inequalities for their solutions in Sections 4 and 5.

2 Carnot Groups

A Carnot group is a connected, simply connected, nilpotent Lie group G of topological dim$G = N \geq 2$ equipped with a graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ so that $[V_1, V_i] = V_{i+1}$ for i=1,2,...,r-1 and $[V_1, V_r] = 0$. This defines an r-step Carnot group. As usual, elements of $\mathcal{G}$ will be identified with left-invariant vectors fields on G.

We fix a left-invariant Riemannian metric g on G with $g(X_i, X_j) = \delta_{ij}$. We denote the inner product with respect to this metric, as well as all other inner products, by $\langle , \rangle$. We assume that $\dim V_1 = m \geq 2$ and fix an orthonormal basis of $V_1 : X_1, X_2, ..., X_m$. The horizontal tangent bundle of G, HT , is the subbundle determined by V_1 with horizontal tangent space HT_x the fiber span$[X_1(x), ..., X_m(x)]$. We use a fixed global coordinate system as $\exp : \mathcal{G} \to G$ is a diffeomorphism (since G is simply-connected and nilpotent). We extend $X_1, ..., X_m$ to an orthonormal basis $X_1, ..., X_m, T_1, ..., T_{N-m}$ of $\mathcal{G}$. All integrals will be with respect to the bi-invariant Harr measure on G which arises as the push-forward of the Lebesque measure in $\mathbb{R}^N$ under the exponential map. We denote by $|E|$ the measure of a measurable set E. We normalize the Harr measure so that the measure of the unit ball is one. We denote by Q the homogeneous dimension of the Carnot group G defined by $Q = \sum_{i=1}^{r} i \dim V_i$. The dual basis of $\mathcal{G}$ is denoted by dx_1,...,dx_m,τ_1,...,τ_{N-m}

We write $|v|^2 = \langle v, v \rangle$, d for the distributional exterior derivative and δ for the codifferential adjoint. We use the following spaces where U is an open set in G :

$C_0^\infty(U)$: infinitely differentiable compactly supported functions in U,

$HW^{1,p}(U)$: horizontal Sobolev space of functions $u \in L^p(U)$ such that the distributional derivatives $X_i u \in L^p(U)$ for $i = 1$,...,m.

When u is in the local horizontal Sobolev space $HW^{1,p}_{loc}(U)$ we write the horizontal differential as $d_0 u = X_1 u dx_1 + ... + X_m u dx_m$. (The horizontal gradient $\nabla_0 u = X_1 u X_1 + ... + X_m u X_m$ appears in the literature. Notice that $|d_0 u| = |\nabla_0 u|$.) The family of dilations on G, $\{\delta_t : t > 0\}$, is the lift to G of the automorphism δ_t of $\mathcal{G}$ which acts on each V_i by multiplication by t^i.

A path in G is called horizontal if its tangents lie in V_1. The (left-invariant) Carnot-Carathéodory distance , $d_c(x, y)$, between x and y is the infimum of the lengths, measured in the Riemannian metric g, of all horizontal paths which join x to y. A homogeneous norm is given by $|x| = d_c(0, x)$. We have $|\delta_t(x)| = t|x|$.

We write $B(x,r) = \{y \in G : |x^{-1}y| < r\}$ for the ball centered at x of radius r. Since the Jacobian determinant of the dilatation δ_r is r^Q and we have normalized the measure, $|B_r| = r^Q$. For $\sigma > 0$, we write σB for the ball with the same center as B and radius σ times that of B. For information about Carnot groups see [5],[6] and [15].

Example 2.1 *Euclidean space* $\mathbb{R}^n$ *with its usual Abelian group structure is a Carnot group. Here* $Q = n$ *and* $X_i = \partial/\partial x_i$.

Example 2.2 *Each Heisenberg group* H_n, $n \geq 1$, *is homeomorphic to* $\mathbb{R}^{2n+1}$. *They form a family of noncomutative Carnot groups which arise as the boundaries of complex n-dimensional hyperbolic space. Denoting points in* H_n *by* (z,t) *with* $z = (z_1, ..., z_n) \in \mathbb{C}^n$ *and* $t \in \mathbb{R}$ *we have the group law given as*

$$(z,t) \circ (z',t') = (z + z', t + t' + 2\sum_{j=1}^{n} Im(z_j \bar{z}'_j)). \tag{1}$$

With the notation $z_j = x_j + iy_j$, *the horizontal space* V_1 *is spanned by the basis*

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \tag{2}$$

$$Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}. \tag{3}$$

The one dimensional center V_2 *is spanned by the vector field* $T = \partial/\partial t$ *with commutator relations* $[X_j, Y_j] = -4T$. *All other brackets of* $\{X_1, Y_1, ..., X_n, Y_n\}$ *are zero. The homogeneous dimension of* H_n *is* $Q = 2n + 2$. *A homogeneous norm is given by*

$$|(z,t)| = (|z|^4 + t^2)^{1/4}. \tag{4}$$

3 Domains in metric spaces

In this section (X, d) is a general metric space, $\Omega \subset X$ is an open bounded proper subset and $\alpha < 0$.

Definition 3.1 *The domain Ω is a δ-John domain if there exists a constant $\delta > 0$ such that each pair of points x_1, x_2 can be joined by a curve $\gamma \subset \Omega$, such that*

$$\min_{j=1,2} l(\gamma(x_j, y)) \leq \delta d(y, \Omega^c)$$

for all $y \in \gamma$. Here $\gamma(x_j, y)$ is a subcurve of γ joining x_j and y. The domain Ω is a John domain if it is a δ-John domain for some δ.

A metric ball $B = B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ in Ω satisfies a *weak geodesic condition* it for every $x \in B$ there exists a curve $\gamma : [0, l] \to B$ of length less than r for which $\gamma(0) = x$ and $\gamma(l) = x_0$. Balls with this property are John domains [4]. In particular, metric balls in Carnot groups satisfy a weak geodesic condition and as such are John domains [4].

We use the seminorms for $f : \Omega \to \mathbb{R}$:

$$O_{\alpha,\Omega}(f) = \inf\{m : |f(x_1) - f(x_2)| \leq m(\min_{j=1,2} d(x_j, \Omega^c))^\alpha, x_1, x_2 \in \Omega\},$$

$$O^{loc}_{\alpha,\Omega}(f) = \inf\{m : |f(x_1) - f(x_2)| \leq m(\min_{j=1,2} d(x_j, \Omega^c))^\alpha, x_1, x_2 \in \Omega,$$

$$d(x_1, x_2) \leq \frac{1}{2} d(x_1, \Omega^c)\}$$

We write $f \in \mathrm{Ord}_\alpha(\Omega)$ when $O_{\alpha,\Omega}(f)$ is finite and $f \in \mathrm{locOrd}_\alpha(\Omega)$ when $O^{loc}_{\alpha,\Omega}$ is finite.

Definition 3.2 *A domain Ω is an Ord_α-extention domain if there exists a constant $\eta > 0$ such that*

$$O_{\alpha,\Omega}(f) \leq \eta O^{loc}_{\alpha,\Omega}(f)$$

for all $f : \Omega \to \mathbb{R}$.

The following results are generalizations of theorems given in [9] for $\mathbb{R}^n$. The proofs in [9] generalize in a straight forward manner to the metric setting given the assumptions on metric balls indicated below.

Theorem 3.3 *If there exists a constant M such that each $x_1, x_2 \in \Omega$ can be joined by a rectifiable curve $\gamma \subset \Omega$ with*

$$\int_\gamma \frac{ds}{d(x, \Omega^c)^{1-\alpha}} \leq M(\min_{j=1,2} d(x_j, \Omega^c))^\alpha, \tag{5}$$

then Ω is an Ord_α-extension domain. If in addition metric balls are John domains, then the converse holds.

Theorem 3.4 *If there exist constants M and $\alpha < 0$ such that each $x_1, x_2 \in \Omega$ can be joined by a rectifiable curve $\gamma \subset \Omega$ with*

$$\int_\gamma \frac{ds}{d(x,\Omega^c)^{1-\alpha}} \leq M(\min_{j=1,2} d(x_j,\Omega^c))^\alpha,$$

then Ω is a John domain. If metric balls satisfy a weak geodesic condition, then the converse holds.

Notice that it follows that if Ω is a John domain and balls satisfy a weak geodesic condition, then Ω is an Ord_α-extension domain for all $\alpha < 0$. Hence John domains in Carnot groups have these extension properties.

Theorem 3.5 *If balls satisfy a weak geodesic condition in Ω and if there exists $c > 0$ such that*

$$\inf\{m : |f(x_1)-f(x_2)| \leq m(\min_{j=1,2} d(x_j,\Omega^c))^\alpha, x_1, x_2 \in \Omega, d(x_1,x_2) \leq cd(x_1,\Omega^c)\}$$

is finite, then $f \in Ord_\alpha(B)$ for all balls $B \subset \Omega$.

Proof: Theorem 3.4 shows that balls in Ω satisfy (5). The theorem then follows by adapting the proof of Theorem 3.3 from [9].

4 Subelliptic A-Harmonic equations

We consider weak solutions to equations of the form

$$\delta A(x,u,d_0u) = B(x,u,d_0u) \tag{6}$$

where $u \in HW^{1,p}(\Omega)$ and $A : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ are measurable and for some $p > 1$ satisfy the structural equations :

$$|A(x,u,\xi)| \leq a_0|\xi|^{p-1} + (a_1(x)|u|)^{p-1},$$
$$\xi \cdot A(x,u,\xi) \geq |\xi|^p - (a_2(x)|u|)^p,$$
$$|B(x,u,\xi)| \leq b_1(x)|\xi|^{p-1} + (b_2(x))^p|u|^{p-1}$$

with $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. Here $a_0 > 0$ and $a_i(x), b_i(x), i = 1,2$, are measurable and nonnegative and are assumed to belong to certain subspaces of

$L^t(\Omega)$, where $t = \max(p, Q)$. See [10]. We refer to these quantities as the structure constants.

A weak solution to (6) means that

$$\int_\Omega \{\langle A(x, u, d_0 u), d_0 \phi \rangle - \phi B(x, u, d_0 u)\} dx = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

We use the exponent $p > 1$ for this purpose throughout the rest of this paper. We refer to a solution of (6) as A-harmonic. Moreover, we also assume from here on that u is A-harmonic in $\Omega \subset G$ with exponent p. We may assume that u is a continuous representative.

For example when $A(x, \xi) = \xi$ and $B(x, \xi) = 0$ the A-harmonic equation is the subLaplacian

$$\delta d_0 u = 0.$$

When $A(x, \xi) = |\xi|^{q-2}\xi, q > 1$, we have the q-subLaplacian

$$\delta |d_0 u|^{q-2} d_0 u = 0.$$

5 A-harmonic Inequalities

Inequality (7) is called the Caccioppoli inequality.

Theorem 5.1 *There exists a constant C , depending only on Q, σ and the structure constants, such that*

$$\int_B |d_0 u|^p \leq C|B|^{-p/Q} \int_{\sigma B} |u - c|^p \tag{7}$$

for all balls B with $\sigma B \subset \Omega$ and any constant c.

See [6]

Theorem 5.2 is a Poincaré-Sobolev inequality, [3], [7] and [6].

Theorem 5.2 *Let $0 < s < \infty$. There exists a constant C, depending only on Q, s and the structure constants, such that*

$$\int_B |u - u_B|^s \leq C|B|^{s/Q} \int_B |d_0 u|^s \tag{8}$$

for all balls $B \subset \Omega$.

Next is a quasi-sub-mean-value property.

Theorem 5.3 *Let $s > p - 1$, $\sigma > 1$. There exists a constant C, depending only on s, Q, σ and the structure constants, such that*

$$|u(x) - c| \leq C(\frac{1}{|B|} \int_{\sigma B} |u - c|^s)^{1/s} \tag{9}$$

for all balls B with $\sigma B \subset \Omega$, all $x \in B$ and any constant c.

See [3].

We also have weak reverse Hölder inequalities for solutions and their horizontal gradients.

Theorem 5.4 *Let $0 < s, t < \infty$. There exists a constant C, depending only on s, t, Q, σ and the structure constants, such that,*

$$(\frac{1}{|B|} \int_B |u - u_B|^t)^{1/t} \leq C(\frac{1}{|B|} \int_{\sigma B} |u - u_B|^s)^{1/s}. \tag{10}$$

for all balls B with $\sigma B \subset \Omega$.

Proof : Integrating (9) we obtain

$$(\frac{1}{|B|} \int_B |u - u_B|^t)^{1/t} \leq C(\frac{1}{|B|} \int_{\sqrt{\sigma} B} |u - u_B|^s)^{1/s} \tag{11}$$

for $s > p - 1$ and all $t > 0$. When $t > s$ this is a weak reverse Hölder inequality which can be improved to the result [8], [3]. Hölder's inequality gives all $0 < s, t < \infty$.

Theorem 5.5 *There exists an exponent $p' > p$, depending only on Q and the structure constants, and there exists a constant C, depending only on Q, s, σ and the structure constants, such that*

$$(\frac{1}{|B|}\int_B |d_0 u|^{p'})^{1/p'} \leq C(\frac{1}{|B|}\int_{\sigma B} |d_0 u|^s)^{1/s} \tag{12}$$

for $s > 0$ and all balls B with $\sigma B \subset \Omega$.

Proof : We combine the Caccioppoli estimate (7), equation (10) and the Poincaré-Sobolev inequality (8),

$$\begin{aligned}(\frac{1}{|B|}\int_B |d_0 u|^p)^{1/p} &\leq C|B|^{-1/Q}(\frac{1}{|B|}\int_{\sqrt{\sigma}B} |u - u_{\sqrt{\sigma}B}|^p)^{1/p} \\ &\leq C|B|^{-1/Q}\frac{1}{|B|}\int_{\sigma B} |u - u_{\sigma B}| \\ &\leq C\frac{1}{|B|}\int_{\sigma B} |d_0 u|.\end{aligned}$$

This is a reverse Hölder inequlity. As such it improves to all positive exponents on the right hand side and to some exponent $p' > p$ on the left. See [8] and [6].

Theorem 5.6 *Let $0 \leq s < \infty$. There is a constant C, depending only on s, Q, σ and the structure constants, such that*

$$osc(u, B) \leq C|B|^{(s-Q)/sQ}(\int_{\sigma B} |d_0 u|^s)^{1/s} \tag{13}$$

for all balls B with $\sigma B \subset \Omega$.

Proof : Fix B with $\sigma B \subset \Omega$ and $x, y \in B$. Using (9) with $s = p$, the Poincaré-Sobolev inequality (8) and (12),

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_{\sqrt{\sigma}B}| + |u(y) - u_{\sqrt{\sigma}B}| \\ &\leq C(\frac{1}{|B|}\int_{\sqrt{\sigma}B} |u - u_{\sqrt{\sigma}B}|^p)^{1/p} \\ &\leq C|B|^{(p-Q)/pQ}(\int_B |d_0 u|^p)^{1/p} \\ &\leq C|B|^{(s-Q)/sQ}(\int_{\sigma B} |d_0 u|^s)^{1/s}.\end{aligned}$$

6 Growth of A-harmonic functions

We assume that u is A-harmonic in a domain Ω in a Carnot group G.

We use the notation, for $x \in \Omega$ and $u : \Omega \to \mathbb{R}$,

$$D_u(x) = (\frac{1}{|B|} \int_B |d_0 u|^p)^{1/p}$$

where $B = B(x, \frac{1}{2} d_c(x, \Omega^c))$.

Theorem 6.1 *The following are equivalent :*

a) There exists a constant C, independent of u, such that

$$D_u(x) \leq C d_c(x, \Omega^c)^{\alpha - 1},$$

for all $x \in \Omega$.

b) $u \in locOrd_\alpha(\Omega)$.

Proof : Assume a). Let $B = B(x_1, \frac{1}{4} d_c(x_1, \Omega^c))$ and $x_2 \in B$. Using (13),

$$|u(x_1) - u(x_2)| \leq C|B|^{(p-Q)/pQ} (\int_{2B} |d_0 u|^p)^{1/p}$$

$$= C|B|^{1/Q} D_u(x_1)$$

$$\leq C|B|^{1/Q} d_c(x_1, \Omega^c)^{\alpha - 1}$$

$$= C d_c(x_1, \Omega^c)^{\alpha}.$$

Since $\frac{1}{2} d_c(x_1, \Omega^c) \leq d_c(x_2, \Omega^c) \leq \frac{3}{2} d_c(x_1, \Omega^c)$ we have $u \in locOrd_\alpha(\Omega)$ using Theorem 3.5.

Conversely, assume b). Using the Caccioppoli estimate (7),

$$D_u(x_1) = |B|^{-1/p} (\int_{2B} |d_0 u|^p)^{1/p}$$

$$\leq C|B|^{-(p+Q)/pQ} (\int_{4B} |u - u(x_1)|^p)^{1/p}$$

$$\leq Cd_c(x_1, \Omega^c)^{\alpha-1}.$$

Theorem 6.2 *Suppose that $\Omega \subset G$ is a John domain. For $\alpha < 0$ the following are equivalent :*

a) There exists a constant C, independent of u, such that

$$D_u(x) \leq Cd_c(x, \Omega^c)^{\alpha-1}$$

for all $x \in \Omega$

b) $u \in Ord_\alpha(\Omega)$.

Along with assuming that u is an A-harmonic function in $\Omega \subset G$ with exponent p we now also assume that v is A-harmonic in Ω with exponent q. We also assume that

$$(\int_B |d_0 u|^p) = (\int_B |d_0 v|^q) \tag{14}$$

for all balls $B \subset \Omega$.

We list some examples.

1. If $f = u + iv$ is analytic in a domain Ω in the complex plane $\mathbb{C}$, then u and v are harmonic, $p = q = 2$ and $|du| = |dv|$.

2. If $f = (f_1, ..., f_m, ..., f_N)$ is quasiregular in $\Omega \subset G$, then the first m components are A-harmonic, $p = Q$ and $|d_0 f_i| = |d_0 f_j|$ a.e. for $i, j = 1, ..., m$. See [6] and [13].

3. We call u p-harmonic in a domain $\Omega \subset \mathbb{C}$ if it is a solution to the p-harmonic equation

$$div|\nabla u|^{p-2}\nabla u = 0$$

in Ω. Its conjugate in the plane is a q-harmonic function v which satisfies

$$|\nabla u|^{p-2}|\nabla u| = (\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x})$$

with $\frac{1}{p} + \frac{1}{q} = 1$. We call such a pair conjugate p,q-harmonic. In this case $|du|^p = |dv|^q$ See [1],[2].

Theorem 6.3 *Suppose that $u \in locOrd_\alpha(\Omega), \alpha < 0$ and $p(\alpha - 1) = q(\beta - 1)$. If $\alpha < 1 - q/p$, then*

$$v \in locOrd_\beta(\Omega).$$

If $\alpha = 1 - q/p$, then v is of bounded mean oscillation order of growth.

Proof : Using (14) and Theorem 6.1,

$$\left(\frac{1}{|B|}\int_B |d_0 v|^q\right)^{1/q} = \left(\frac{1}{|B|}\int_B |d_0 u|^p\right)^{1/q}$$

$$\leq C d_c(x,\Omega^c)^{(\alpha-1)p/q}$$

$$= C d_c(x,\Omega^c)^{\beta-1}.$$

If $\alpha < 1 - q/p$, then it follows from Theorem 6.1 that $v \in locOrd_\beta(\Omega)$. If $\alpha = 1 - q/p$, then $\beta = 0$. In this case with the use of (8) and (12), for a ball $B = B(x, \frac{1}{2}d_c(x,\Omega^c))$

$$\frac{1}{|B|}\int_B |v - v_B| \leq C|B|^{(1-Q)/Q}\int_B |d_0 v|$$

$$\leq C|B|^{1/Q}\left(\frac{1}{B}\int_B |d_0 v|^q\right)^{1/q}$$

$$\leq C|B|^{1/Q} d_c(x,\Omega^c)^{(\alpha-1)p/q} \leq C.$$

Metric balls in homogeneous groups are John domains and satisfy a Boman chain condition. This condition allows the patching together of local L^p-inequalities into global L^p-inequalities. See [3],[4],[8] and [13]. Hence v is of bounded mean oscillation in Ω.

Theorem 6.4 *If $\Omega \subset G$ is a John domain, $\alpha < 1 - q/p$ and $u \in Ord_\alpha(\Omega)$, then $v \in Ord_\beta(\Omega)$ where $p(\alpha - 1) = q(\beta - 1)$.*

We list some examples in the plane which illustrate these theorems.

The polar angle θ is p-harmonic for all $1 < p < \infty$ in the domain $\Omega_0 = \{re^{i\theta} | r > 0, -\pi < \theta < \pi\}$.

The quasi-radial conjugate p, q-harmonic functions are described in [1] and [2]. These functions have the form

$$u = r^k f(\theta), v = r^l g(\theta),$$

where $k, l \in \mathbb{R}$, r and θ are the usual polar coordinates in $\mathbb{R}$ and

$$p(k-1) = q(l-1).$$

1. If $p = q = 2$, then u and v are conjugate harmonic functions. Quasi-radial conjugate harmonic functions in Ω_0 are given by $u = r^k \cos k\theta$ and $v = r^k \sin k\theta$. When $k < 0$ we have $u, v \in Ord_k(\Omega_0)$. Conjugate harmonic functions are also simultaneously in BMO or a local Lipschitz class. See [11] and [12].

2. For $p \neq 2$, $u = (p-1)r^{(p-2)/(p-1)}/(p-2)$ and $v = \theta$ are conjugate p, q -harmonic functions in Ω_0. When $p < 2$, $u \in Ord_k(\Omega_0)$ where $k = (p-2)/(p-1)$ while v is of bounded mean oscillation in Ω_0.

3. For $p \neq 2$ there exist conjugate p, q-harmonic functions in Ω_0 of the form

$$u = r^k f(\theta), v = r^l g(\theta)$$

where $f(\theta)$ and $g(\theta)$ are bounded. For $k, l < 0$, $u \in Ord_k(\Omega_0)$ while $v \in Ord_l(\Omega_0)$.

Other ranges of values for α and β involve Lipschitz classes. See [11] and [14].

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